Computer Vision

Mathematics

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Definition[Group]

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Definition[Commutative / Abelian Group]

*A group $(G, \oplus)$ is a **commutative / Abelian group** if and only if:*

- $\forall a, b \in G, \; a \oplus b = b \oplus a$ **Commutativity**

Properties[Group]

Let $(G, \oplus)$ a group where n is the identity element. It can be demonstrated that:

■ $\exists n \in G, \forall a, m \in G, \ (a \oplus m = a) \rightarrow (n = m)$ **Identity uniqueness**

■ $\forall a, b, c \in G, \ (a \oplus b = n \text{ and } a \oplus c = n) \rightarrow (b = c)$ **Inverse elements uniqueness**

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Example[Group]

Integers with addition $(\mathbb{Z}, +)$ is a group.

$\mathbb{Z} = \{\dots, -4, -3, -2, 0, 1, 2, 3, \dots\}$

$n = 0$

Definition[Field]

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■ $(F \setminus \{n\}, \otimes)$ is a commutative group (where n denotes the $\oplus$ identity element)

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Notations[Field]

- $\oplus$ is called addition and 0 denotes its identity (neutral) element

- $\otimes$ is called multiplication and 1 denotes its identity (neutral) element

Example[Field]

The real numbers set $\mathbb{R}$ with classic addition $+$ and multiplication $\times$ is a field.

Let $\mathbb{C}$ the complex numbers set such as $\forall c \in \mathbb{C}, \exists u, v \in \mathbb{R}, \ c = u + vi$ with $i^2 = -1$

The structure $(\mathbb{C}, \oplus, \otimes)$ such as:

- $a \oplus b = (a_u + b_u) + (a_v + b_v)i$
- $a \otimes b = (a_u \times b_u - a_v \times b_v) + (a_u \times b_v + a_v \times b_u)i$

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is a field

$+ \neq +$

Definition[Vector]

*Let F be a set. A **vector** over F is a finite tuple, denoted $\boldsymbol{v = (v_1, \ldots, v_i, \ldots, v_n)}$ such as:*

$$v = (v_1, \ldots, v_i, \ldots, v_n), n \in \mathbb{N}^+, 1 \leq i \leq n, v_i \in F$$

*Each v_i is called **component** (or **value**). The number n is called **size** of the vector.*

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Example[Vector]

Let $\mathbb{Z}$ the integers.

$v = (-12, 1, 3)$ is a vector on $\mathbb{Z}$ of size 3 that contains 3 components: -12, 1 and 3

$v = (0, 0, 0, 0)$ is a vector on $\mathbb{Z}$ of size 4 that contains 4 components, all equals to 0

Definition[Vector space]

*Let $\mathbb{K} = (K, \oplus, \otimes)$ be a field. A **vector space** over $\mathbb{K}$ (or $\mathbb{K}$-**vector space**), denoted E, is a non-empty set of vectors of size n on K with two binary operations $+$ and $\cdot$ that satisfies:*

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■ $\forall \alpha, \beta \in \mathbb{K}, \ \forall x \in E, \ \alpha \cdot (\beta \cdot x) = (\alpha \otimes \beta) \cdot x$ **Compatibility**

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■ $\forall x \in E, \ n \cdot x = x$, where n is the identity element of $\otimes$ **Identity**

Definition[Scalar]

Let E a *vector space* over the *field* $\mathbb{K} = (K, \oplus, \otimes)$. All $v \in E$ is called a vector on K and all $\alpha \in K$ is called a **scalar**. The scalar set can be denoted either K or $\mathbb{K}$

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Notations[Operations]

When dealing with group, field or vector space, operations can share the same denotation with respectively $+$ for the addition and $\cdot$ for the multiplication. The nature of the underlying operation is given by the context and the operands nature.

- For a group, $G(\oplus) \equiv G(+)$ - For a field $F(\oplus, \otimes) \equiv F(+, \cdot)$

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Concision[·]

The multiplication operator can be writing in a concise way with $\boldsymbol{a} \cdot \boldsymbol{b} \equiv \boldsymbol{ab}$

Example[Vector space]

Let $\mathbb{R}$ the field of real numbers. The structure E that contains all the vectors of size n on $\mathbb{R}$ and the binary operations $+$ and $\cdot$ such as:

$$\forall u, v \in E, \ u + v = (u_1 + v_1, \dots, u_i + v_i, \dots, u_n + v_n)$$

$$\forall \alpha \in \mathbb{R}, \ \forall u \in E, \ \alpha \cdot u = (\alpha \cdot u_1, \dots, \alpha \cdot u_i, \dots, \alpha \cdot u_n)$$

Is a vector space

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Is a vector space

$$+ \neq +$$
$$\cdot \neq \cdot$$

Binary operations are different

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Is a vector space   Concise writing

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Definition[Null vector]

Let E a vector space of size n over the field $\mathbb{K}$. The identity (neutral) element for the binary operation $+$ is called **null vector** and is denoted $\mathbf{0}_E$. The value for all components of 0_E is the identity element of the addition for $\mathbb{K}$. More formally:

$$0_E = (0, \ldots, 0, \ldots 0)$$

Definition[Linear combination]

Let E be a *vector space* over the *field* $\mathbb{K}$ and let $V \subseteq E$ be a finite subset of vectors such as $V = \{v_1, \ldots, v_i, \ldots, v_m\}, 1 \leq i \leq m$. A **linear combination** of V, denoted $\mathcal{L}(V)$, is defined such as:

$$\mathcal{L}(V) = \alpha_1 v_1 + \cdots + \alpha_i v_i + \cdots + \alpha_m v_m, \ \alpha_i \in \mathbb{K}$$

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Let E be a vector space over the field $\mathbb{R}$ and let $V \subseteq E$ be a finite subset such as:

$$V = \{v_1, v_2\} \text{ with } v_1 = (3, 5, 4) \text{ and } v_2 = (0, 8, -2)$$

The vector $u = 2v_1 + 5v_2 = (2 \cdot 3, 2 \cdot 5, 2 \cdot 4) + (5 \cdot 0, 5 \cdot 8, 5 \cdot -2)$
$$= (6, 10, 8) + (0, 40, -10) = (6, 50, -2)$$

Is a linear combination of v_1 and v_2.

Definition[Linear independence]

Let E be a vector space over the field $\mathbb{K}$ and let $V \subseteq E$ be a finite subset of vectors such as $V = \{v_1, \ldots, v_i, \ldots, v_m\}, 1 \leq i \leq m$. The vectors from V are **linearly independent** if and only if:

$$\mathcal{L}(V) = \sum_{i=1}^{m} \alpha_i v_i = 0_E \rightarrow \forall i, 1 \leq i \leq m, \alpha_i = 0$$

The only way to have a linear combination of vectors that is equal to the null vector is to have all the scalars set to 0

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Let E be a vector space over the field $\mathbb{R}$ and let $V \subseteq E$ be a finite subset such as $V = \{v_1, v_2\}$ with $v_1 = (3, 0, 0)$ and $v_2 = (0, 1, 1)$.

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We have $\mathcal{L}(V) = \alpha v_1 + \beta v_2 = (3\alpha, 0,0) + (0, \beta, \beta) = (3\alpha, \beta, \beta)$

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We have $\mathcal{L}(V) = \alpha v_1 + \beta v_2 = (3\alpha, 0,0) + (0, \beta, \beta) = (3\alpha, \beta, \beta)$

As there is no scalar $\alpha \neq 0$ and $\beta \neq 0$ that satisfy $\alpha v_1 + \beta v_2 = O_E = (0,0,0)$, vectors v_1 and v_2 are linearly independent.

Definition[Basis]

Let E be a vector space and let $S \subseteq E$ be a finite subset of vectors such as $S = \{s_1, \ldots, s_i, \ldots, s_m\}, 1 \leq i \leq m$. The set S is a **basis** *of E is it satisfies:*

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■ All the vectors from S are linearly independent

■ $\forall v \in E, v = \mathcal{L}(S) = \displaystyle\sum_{i=1}^{m} \alpha_i s_i$ 　　　**Space spanning**

All the vector of E are the result of a linear combination of the vectors of S

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◼ All the vectors from S are linearly independent

◼ $\forall v \in E, v = \mathcal{L}(S) = \sum_{i=1}^{m} \alpha_i s_i$ **Space spanning**

Example[Basis]

Let E be a vector space over $\mathbb{R}$ that contains vectors with a size of 3. The set $S = \{s_1, s_2, s_3\}$ with $s_1 = (1,0,0)$, $s_2 = (0,1,0)$ and $s_3 = (0,0,1)$ is a basis of E

Properties[Basis]

Let E a vector space with a basis $B = \{b_1, \dots, b_i, \dots, b_n\}$. The decomposition of all vector $v \in E$ with respect to B is unique. More formally:

$$\forall u, v \in E, u = \sum_{i=1}^{n} \alpha_i b_i \ \ and \ \ v = \sum_{i=1}^{n} \alpha_i b_i \leftrightarrow u = v$$

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Proof

Definition[Dimension]

*Let E be a vector space. All basis S of E have the same cardinality, denoted n, that is called **dimension** of the vector space (Steinitz exchange lemma).*

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Notation

A vector space of **dimension** n over the field $\mathbb{K}$ can be denoted $\mathbb{K}^n$

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Dimension - Size

All the vectors of a vector space $\mathbb{K}^n$ have a size of n (Steinitz exchange lemma)

Representing space

Physical spaces can be represented using vector spaces
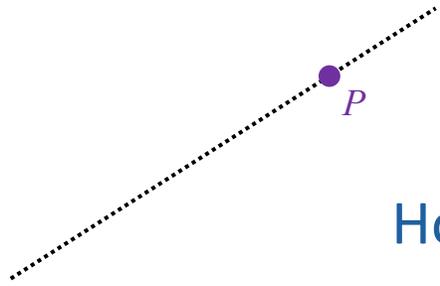
1 dimension

How to express a point P on a line ?

Representing space

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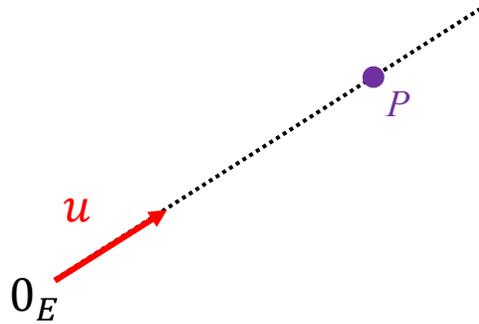
1 dimension

Vector space $\mathbb{K}$
$0_E = (0)$, basis $\{u\}$

Representing space

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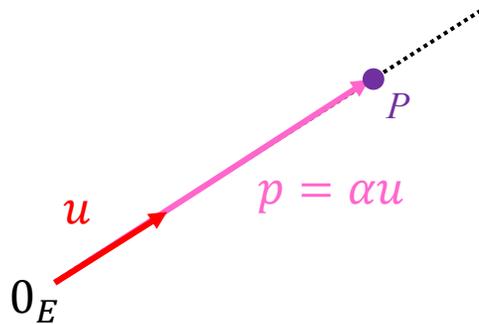
1 dimension

A point P can be assimilated to a vector p

Vector space $\mathbb{K}$
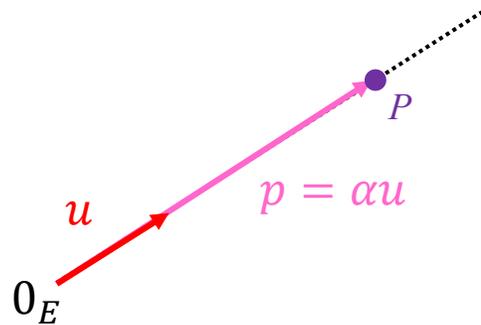$0_E = (0)$, basis $\{u\}$
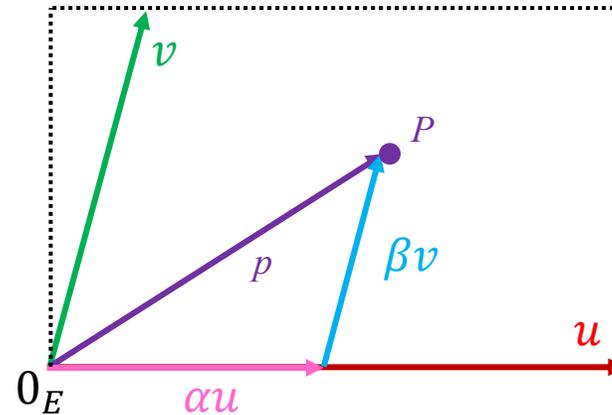$P \equiv p,\ p = \alpha u$

Representing space

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1 dimension

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2 dimensions

Representing space

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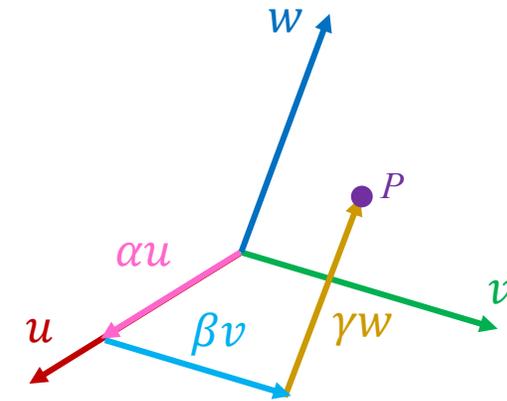
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2 dimensions

Vector space $\mathbb{K}^2$
$0_E = (0,0)$, basis $\{u, v\}$

Representing space

Physical spaces can be represented using vector spaces

1 dimension

2 dimensions

A point P can be assimilated to a vector p

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$0_E = (0,0)$, basis $\{u, v\}$
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Representing space

Physical spaces can be represented using vector spaces

1 dimension	**2 dimensions**

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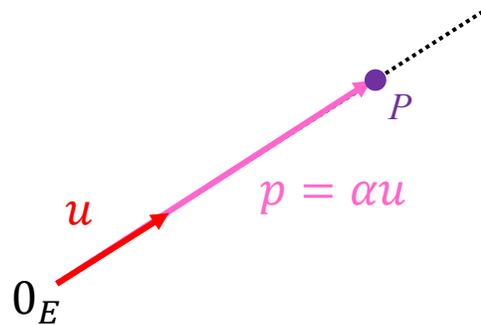
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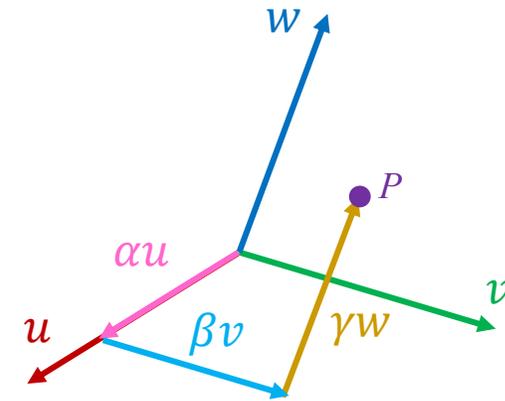| **1 dimension**	**2 dimensions**	**3 dimensions**

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Vector space $\mathbb{K}^3$
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Representing space

Physical spaces can be represented using vector spaces

Infinity of representations
for a given dimension

1 dimension

2 dimensions

3 dimensions
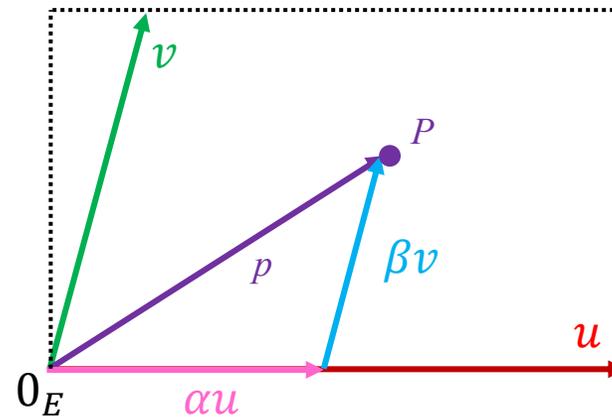
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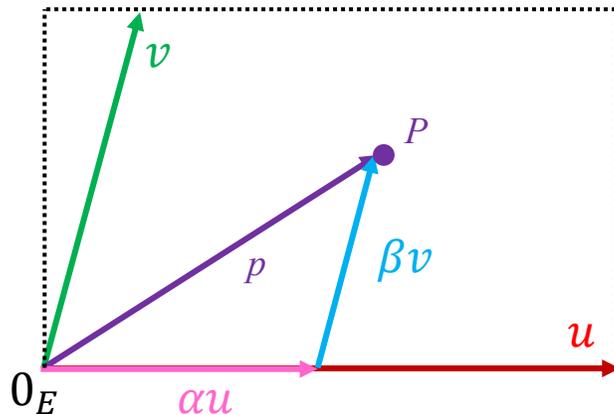$P \equiv p, \; p = \alpha u + \beta v + \gamma w$

Definition[Coordinates, Ordinate]

Let E be a vector space over $\mathbb{K}$ with a finite dimension n, let $B = \{b_1, \dots, b_i, \dots, b_n\}$ be a basis of E. All vector $v \in E$ are such that $v = \alpha_1 b_1 + \cdots + \alpha_i b_i + \cdots + \alpha_n b_n, \; \alpha_i \in \mathbb{K}$. We call **coordinate** of v the tuple $(\boldsymbol{\alpha_1, \dots, \alpha_i, \dots, \alpha_n})$. Each value α_i is called the **ordinate** of index i.

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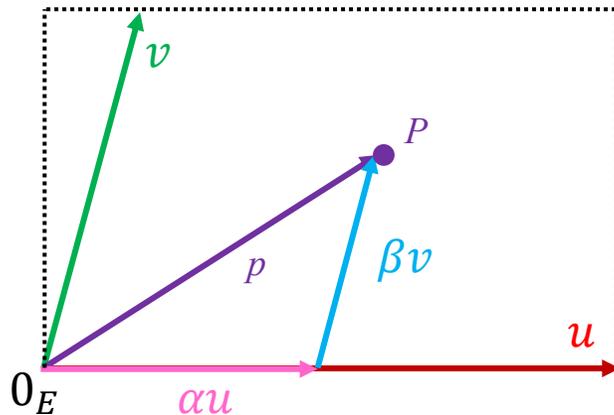
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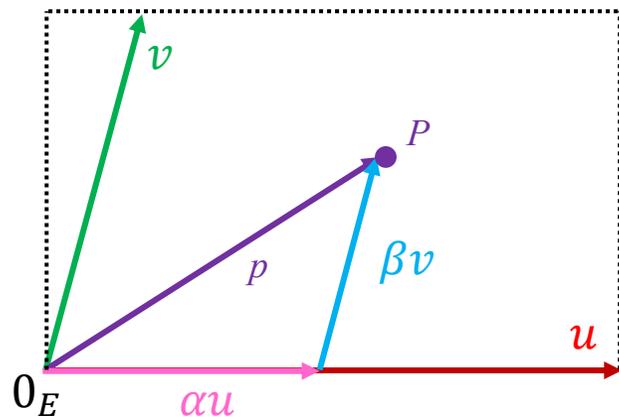$0_E = (0,0)$, basis $\{u, v\}$
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Ordinate of index 1

Ordinate of index 2

Properties[Coordinates]

Let E be a vector space over $\mathbb{K}$ with a finite dimension n, let $B = \{b_1, \ldots, b_i, \ldots, b_n\}$ be a basis of E, the coordinates of all vectors $v \in E$ are unique.

Proof: coordinates $\equiv$ vector decomposition

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Let E be a vector space over $\mathbb{K}$ with a finite dimension n, let $B = \{b_1, \dots, b_i, \dots, b_n\}$ be a basis of E, For all vector $v \in E$ the following notations are equivalent:

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Vector notation

Coordinates notation
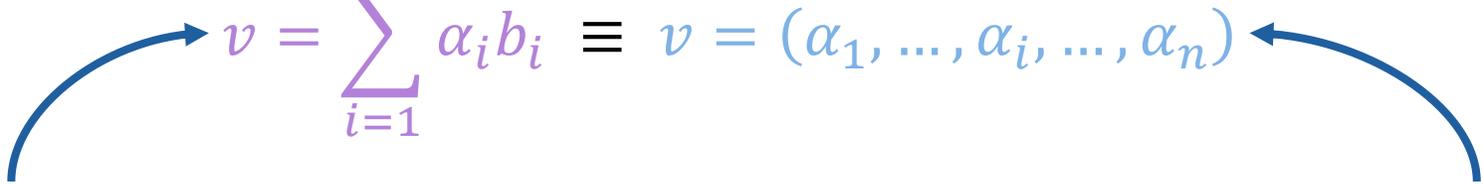
Representing space

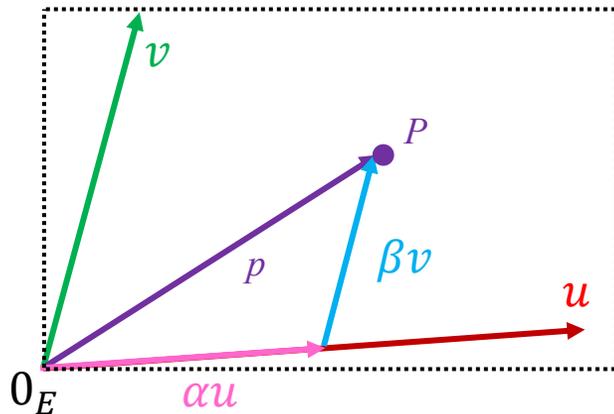Physical spaces can be represented using vector spaces

2 dimensions

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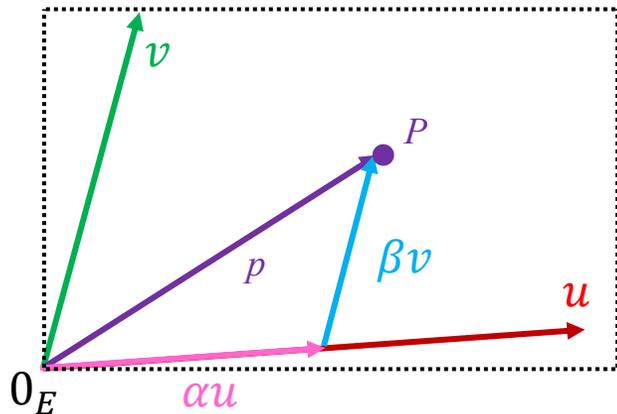Physical spaces can be represented using vector spaces

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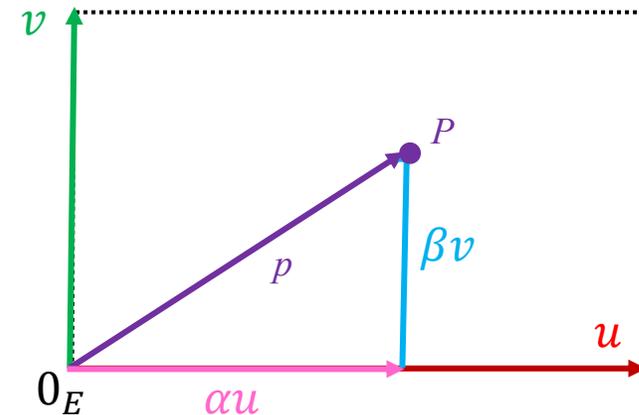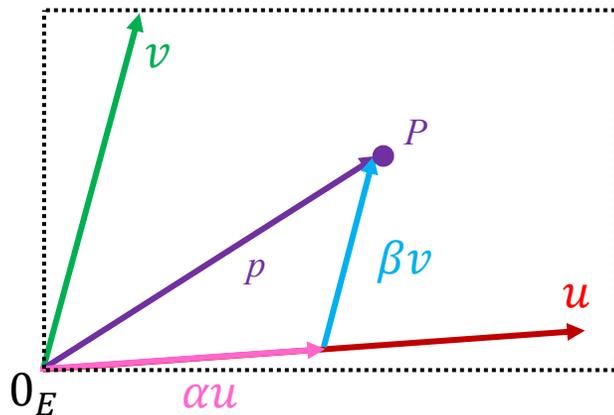Physical spaces can be represented using vector spaces

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Definition[Dot product]

*Let E be a vector space over $\mathbb{K}$ of finite dimension n. We call **dot product**, denoted $\cdot$, the bilinear map defined such as:*

$$\forall u, v \in E, u = \sum_{i=1}^{n} \alpha_i b_i, v = \sum_{i=1}^{n} \beta_i b_i, \ u \cdot v = \sum_{i=1}^{n} \alpha_i \beta_i$$

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Example[Dot product]

Let E be a vector space over $\mathbb{R}$ of dimension 2 together with the dot product. Let u and v two vectors such as $u = (\alpha_1, \alpha_2) = (2,3)$ and $v = (\beta_1, \beta_2) = (4,1)$, we have:

$$u \cdot v = \sum_{i=1}^{2} \alpha_i \beta_i = \alpha_1 \beta_1 + \alpha_2 \beta_2 = 2 \times 4 + 3 \times 1 = 8 + 3 = 11$$

Properties[Dot product]

Let E be a vector space over $\mathbb{K}$ of finite dimension n together with the dot product $\cdot$. For all vectors $u, v \in E$, $u = (\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)$, $v = (\beta_1, \ldots, \beta_i, \ldots, \beta_n)$, $1 \leq i \leq n$ we have:

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Addition and multiplication from $\mathbb{K}$

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■ $u \cdot u = \displaystyle\sum_{i=1}^{n} \alpha_i \alpha_i = 0 \longrightarrow u = 0_E$

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$\quad\quad \alpha_i \alpha_i = 0 \longrightarrow \alpha_i = 0$ and so $u = (\alpha_1, \dots, \alpha_i, \dots, \alpha_n) = (0, \dots, 0) = 0_E$

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Definition[Euclidean space]

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An **Euclidean space**, denoted E^n or $\mathbb{R}^n$ is a *vector space* over $\mathbb{R}$ of finite *dimension* n together with the *dot product*.

Definition[Euclidean norm]

Let E^n be an *Euclidean space* and let $u \in E^n$ a *vector*. The **Euclidean norm** of u, denoted $\|u\|$, is defined such as:

$$\|u\| = \sqrt{u \cdot u} = \sqrt{\sum_{i=1}^{n} {\alpha_i}^2}$$

Definition[Orthogonal basis]

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Let E^3 be an Euclidean space and let $B = \{b_1, b_2, b_3\}$ its basis with $b_1 = (0,2,0)$, $b_2 = (3,0,0)$ and $b_3 = (0,0,4)$. We have:

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$b_2 \cdot b_3 = 3 \times 0 + 0 \times 0 + 0 \times 4 = 0 = b_3 \cdot b_2$ Symmetry of the dot product $\cdot$

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$\forall b_i, b_j \in B, 1 \leq i, j \leq n, \ b_i \cdot b_j = 0$, the basis B is orthogonal.

Representing space

Physical spaces can be represented using vector spaces

2 dimensions

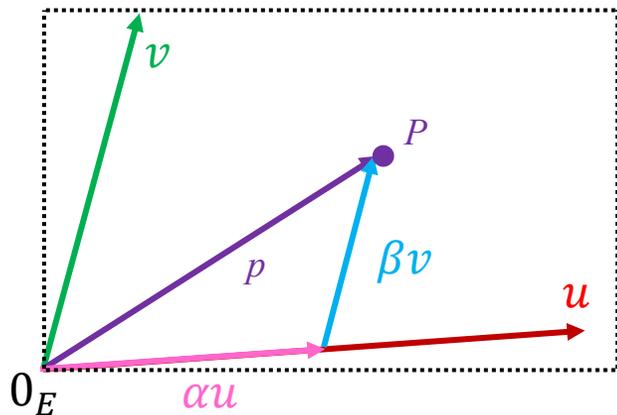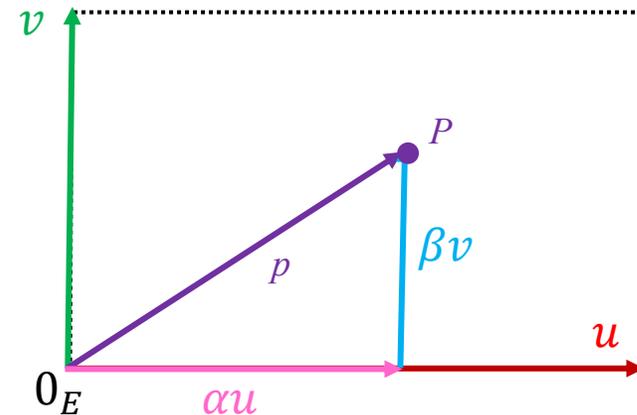Vector space $\mathbb{K}^2$
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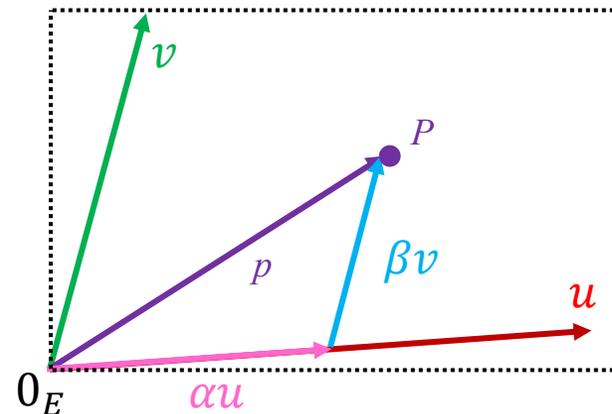
Representing space

Physical spaces can be represented using Euclidean spaces

2 dimensions

Euclidean space E^2
$0_E = (0,0)$
Orthogonal basis: $\{u, v\}$
$u = (2,0), v = (0,3)$

Vector space $\mathbb{K}^2$
$0_E = (0,0)$, basis $\{u, v\}$

Can we go further ?

Definition[Orthonormal basis]

*Let E^n an Euclidean space. Its basis $B = \{b_i\}, 1 \leq i \leq n$, is **orthonormal** if and only if B is orthogonal and all the vectors $b_i \in B$ have an Euclidean norm of 1. More formally:*

$$\forall b_i, b_j \in B, 1 \leq i, j \leq n, \ \ b_i \cdot b_j = 0 \ \ and \ \ \|b_i\| = \|b_j\| = 1$$

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Definition[Unit vector]

*Let E^n an Euclidean space with an orthonormal basis $B = \{b_i\}, 1 \leq i \leq n$. Each vector of $b_i \in B$ has an Euclidean norm of 1 and is called **unit vector** for dimension i.*

Example[Orthonormal basis]

Let E^3 be an Euclidean space and let $B = \{b_1, b_2, b_3\}$ its basis with $b_1 = (0,1,0)$, $b_2 = (1,0,0)$ and $b_3 = (0,0,1)$. We have:

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$$b_1 \cdot b_2 = 0 \times 1 + 1 \times 0 + 0 \times 0 = 0 = b_2 \cdot b_1$$

$$b_2 \cdot b_3 = 1 \times 0 + 0 \times 0 + 0 \times 2 = 0 = b_3 \cdot b_2 \qquad B \text{ is orthogonal}$$

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$$b_1 \cdot b_2 = 0 \times 1 + 1 \times 0 + 0 \times 0 = 0 = b_2 \cdot b_1 \qquad \|b_1\| = \sqrt{b_1 \cdot b_1} = \sqrt{0^2 + 1^2 + 0^2} = 1$$

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$b_i \in B$ have an Euclidean norm of 1

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$$\forall b_i, b_j \in B, 1 \leq i, j \leq n, \ b_i \cdot b_j = 0 \text{ and } \|b_i\| = \|b_j\| = 1$$

the basis B is orthonormal and b_1, b_2 and b_3 are unit vectors.

Theorem[Orthonormality]

Every Euclidean space has an orthonormal basis (*Gram-Schmit process*)

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Proof[Orthonormality]

■ Every Euclidean space E^n has an orthogonal basis (*Gram-Schmit process*)

■ For every orthogonal basis $B = \{b_1, \dots, b_i, \dots, b_n\}$, exists a linear form, denoted p_F, such as:

$$\forall x \in E^n, p_F(x) = \sum_{i=1}^{n} \frac{b_i \cdot x}{\|b_i\|^2} b_i$$

Representing space

Physical spaces can be represented using Euclidean spaces with orthonormal basis

2 dimensions

Euclidean space E^2
$0_E = (0,0)$
Orthogonal basis: $\{u, v\}$
$u = (2,0), v = (0,3)$
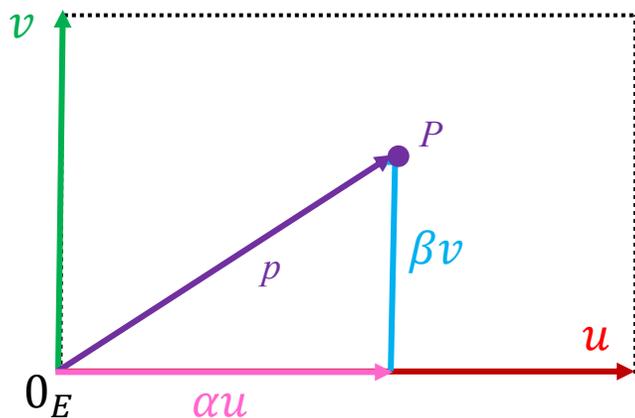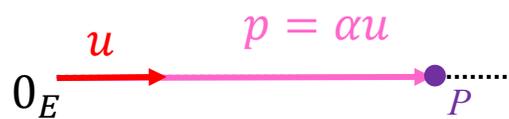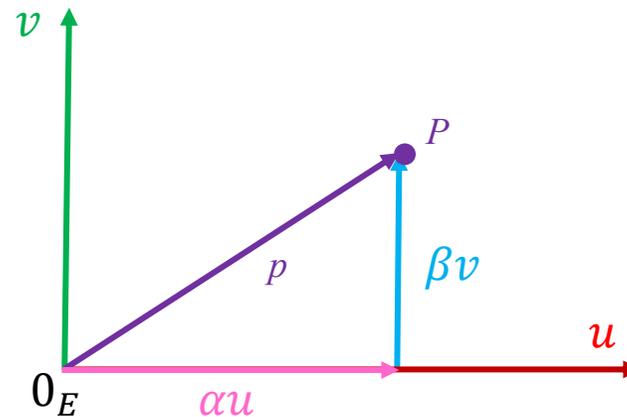
Euclidean space E^2
$0_E = (0,0)$
Orthonormal basis: $\{u, v\}$
$u = (1,0), v = (0,1)$
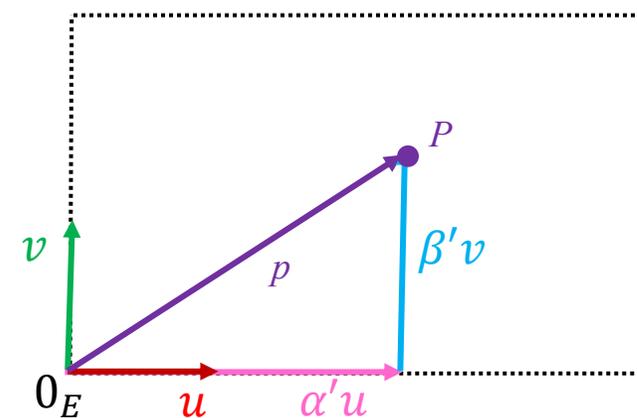
Representing space

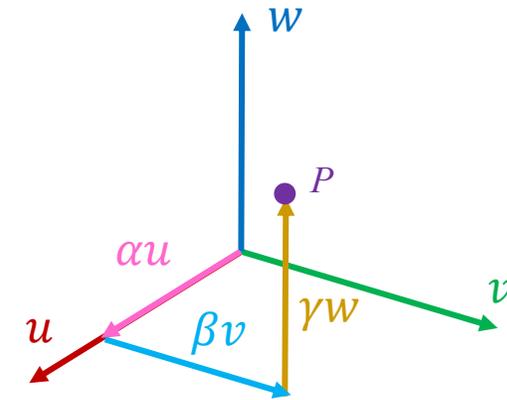Physical spaces can be represented using Euclidean spaces with orthonormal basis

1 dimension

2 dimensions

3 dimensions

Euclidean space E^1, $0_E = (0)$
Orthonormal basis: $\{u\}$
$u = (1)$

Euclidean space E^2
Orthonormal basis: $\{u, v\}$
$u = (1,0), v = (0,1)$

Euclidean space E^3
Orthonormal basis: $\{u, v, w\}$
$u = (1,0,0), v = (0,1,0)$
$w = (0,0,1)$