Computer Vision

Referential & Transforms

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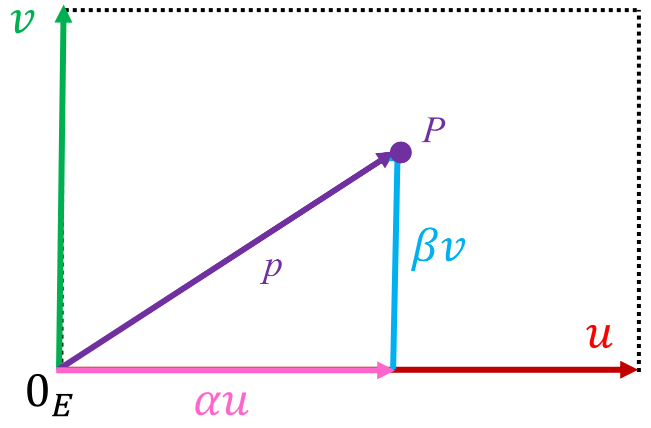
Representing space

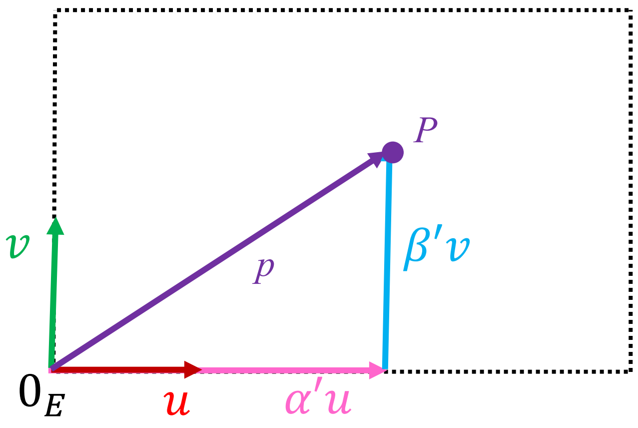
Physical spaces can be represented using Euclidean spaces with orthonormal basis

2 dimensions

Euclidean space E^2
$0_E = (0,0)$
Orthogonal basis: $\{u, v\}$
$u = (2,0), v = (0,3)$



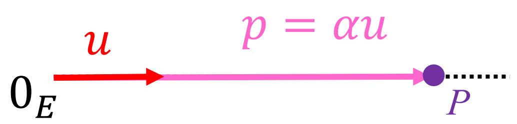
Euclidean space E^2
$0_E = (0,0)$
Orthonormal basis: $\{u, v\}$
$u = (1,0), v = (0,1)$



Representing space

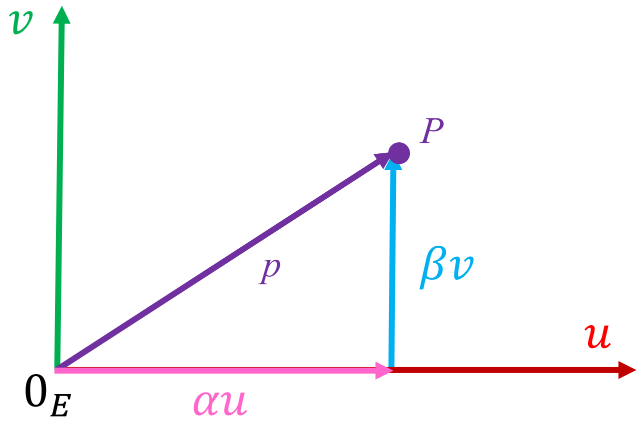
Physical spaces can be represented using Euclidean spaces with orthonormal basis

1 dimension



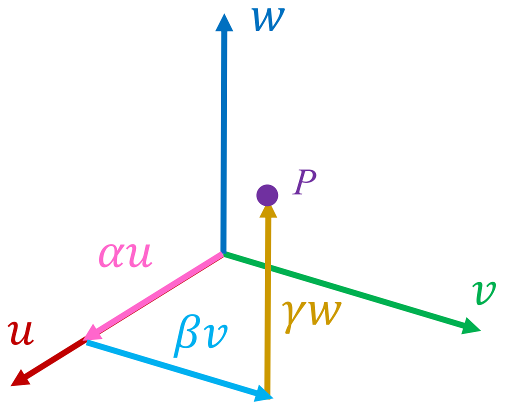
Euclidean space E^1, $0_E = (0)$
Orthonormal basis: $\{u\}$
$u = (1)$

2 dimensions



Euclidean space E^2
Orthonormal basis: $\{u, v\}$
$u = (1,0), v = (0,1)$

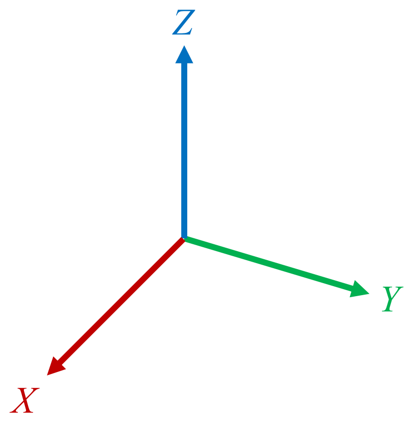
3 dimensions



Euclidean space E^3
Orthonormal basis: $\{u, v, w\}$
$u = (1,0,0), v = (0,1,0)$
$w = (0,0,1)$

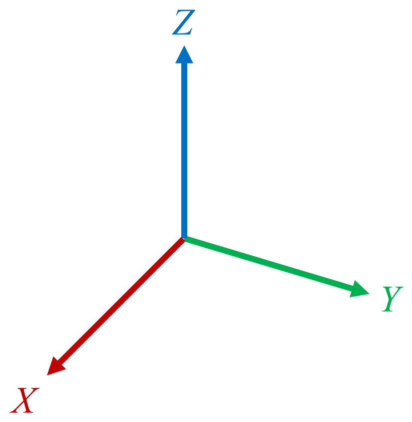
Representing 3D space

- 3D space can be represented with Euclidean Space
  - Vector space of dimension 3
  - Basis of 3 vectors denoted X, Y and Z



Representing 3D space

- 3D space can be represented with Euclidean Space
  - Vector space of dimension 3
  - Basis of 3 vectors denoted X, Y and Z
  - dot product $\cdot$ and cross product $\times$



Representing 3D space

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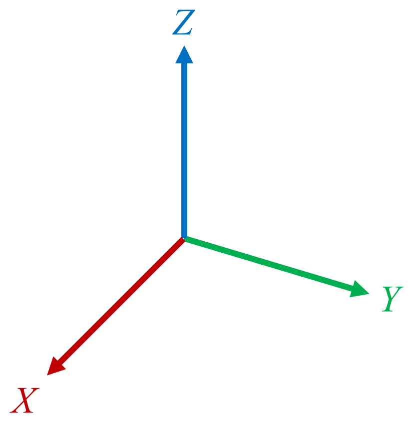


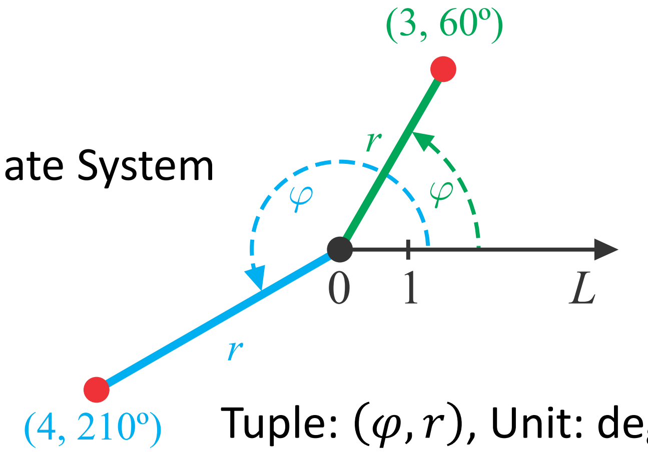
- Mathematical properties
  - Vector orthogonality: $X \cdot Y = X \cdot Z = Y \cdot Z = 0$
  - Plane orthogonality: $X \times Y = Z$, $X \times Z = Y$, $Y \times Z = X$

Definition[Coordinates System]

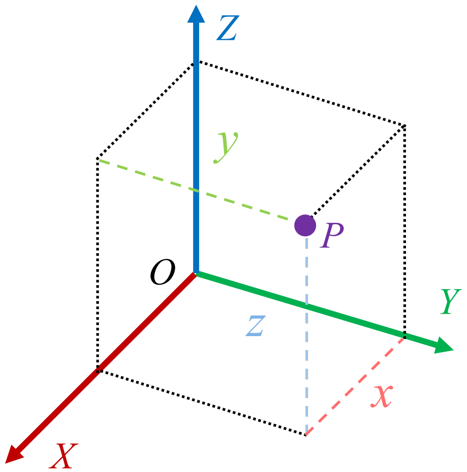
a coordinate system is a system that uses numbers, or coordinates, to uniquely determine the position of geometric elements within a topological space.

- Coordinates are expressed as **tuples**
- Origin is tuple (0)
- Provide **unit length** for all coordinates

Polar Coordinate System



Tuple: (φ, r), Unit: degree (°)



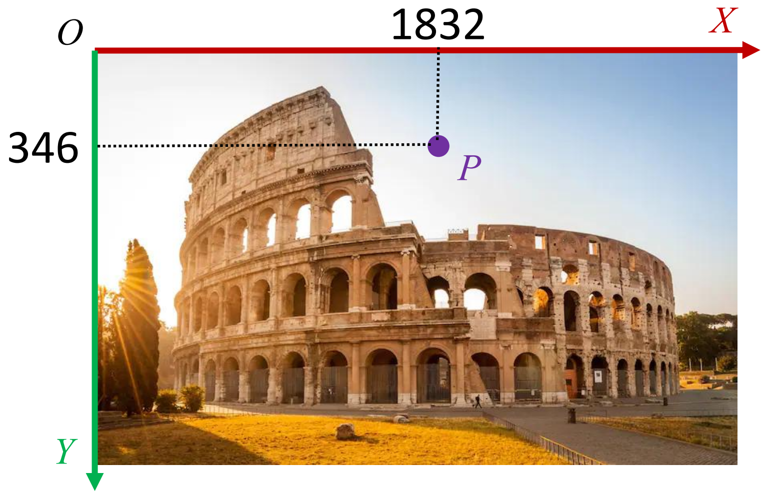
Cartesian Coordinate System

Tuple (x, y, z), Unit: meter (m)

Definition[Referential (Reference system)]

a referential (reference system) is defined by a reference frame, a coordinate system and an origin position.

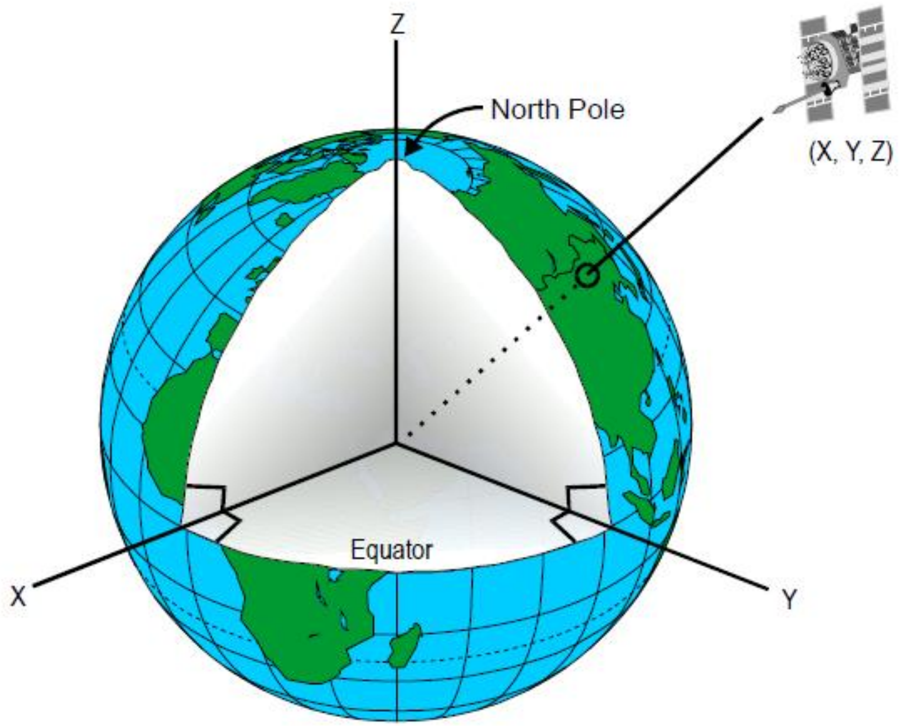
- Example: Image / Screen reference system
  - Reference Frame: X, Y (2D Euclidean space)
  - Coordinate system: Cartesian
  - Coordinate unit: 1 pixel
  - Origin O: The upper left corner



$P = (1832, 346)$

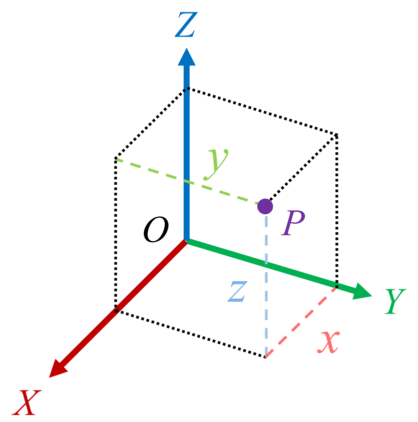
Referential (Reference system)

- Example: Earth-centered, Earth-fixed coordinate system (ECEF)
  - O: The center of mass of the Earth
  - Z axis: on the line from O to North Pole
  - X axis: in the plane of the equator
  - Y axis: in the plane of the equator, 90° from X axis



Representing 3D position

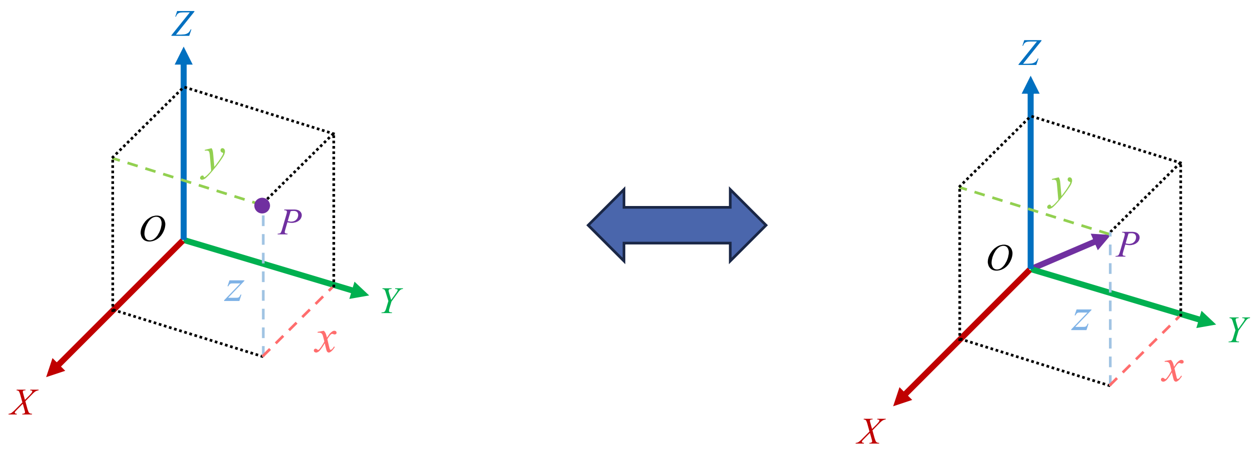
- 3D Positions can be represented using Cartesian Coordinates
  - A position is a tuple (x, y, z)
  - The origin of the coordinate system is denoted O
  - Coordinates of point P are denoted (p_x, p_y, p_z)



- Properties
  - A tuple (p_x, p_y, p_z) describes one and only one point
  - Coordinate p_i of a point P is the distance between P and its orthogonal projection onto the orthogonal plane to the i axis

Representing 3D position

- 3D Positions can be represented as a vector



$$P = P - (0,0,0) = \overrightarrow{OP}$$

- The position $P = (x, y, z)$ of an object within a 3D space is equivalent to the vector $\overrightarrow{OP}$, where O is the origin of the space

Translation

- Let $P = (x, y, z)$ a point and (α, β, γ) a vector. A translation, denoted $T(\alpha, \beta, \gamma)$, is an application such as:

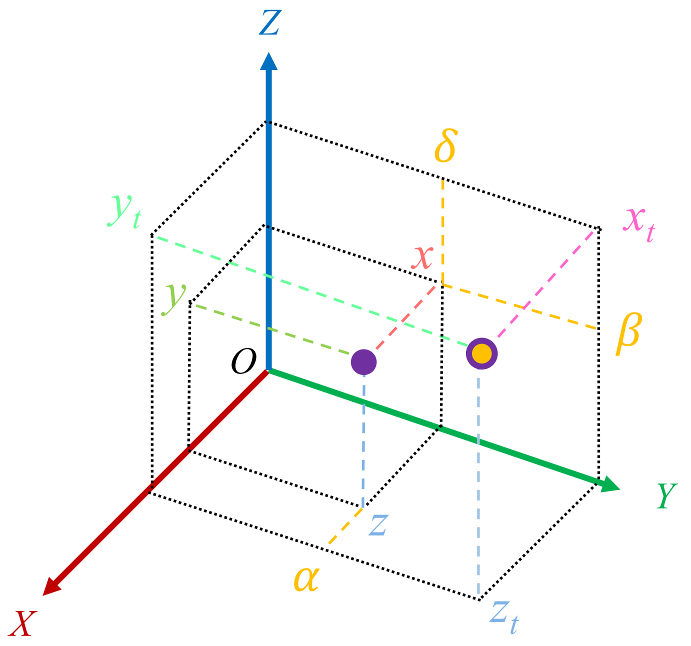
$$T(\alpha, \beta, \gamma)(P) = (x_t, y_z, z_t)$$

- With:

$$x_t = x + \alpha$$

$$y_t = y + \beta$$

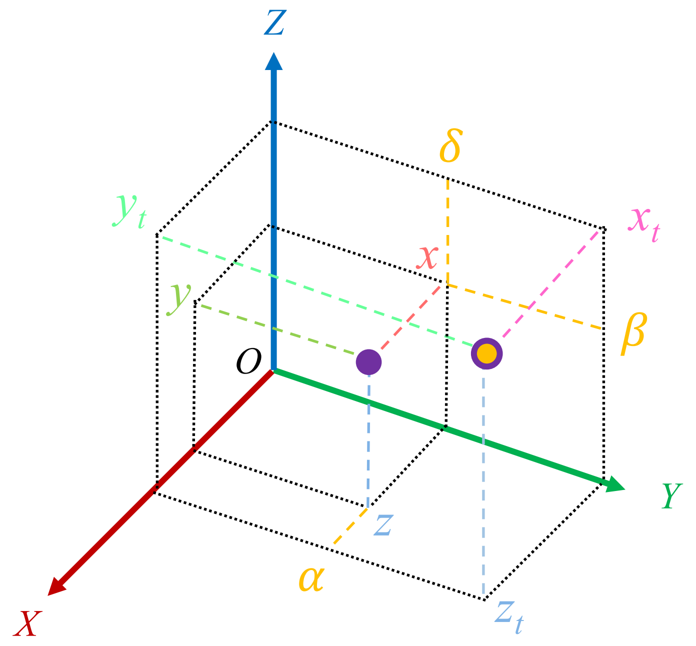
$$z_t = z + \delta$$



Translation properties

- Let $P = (x, y, z)$ be a point, $\Theta = (\alpha, \beta, \gamma)$ be a vector and $T(\alpha, \beta, \gamma)$ be a translation:

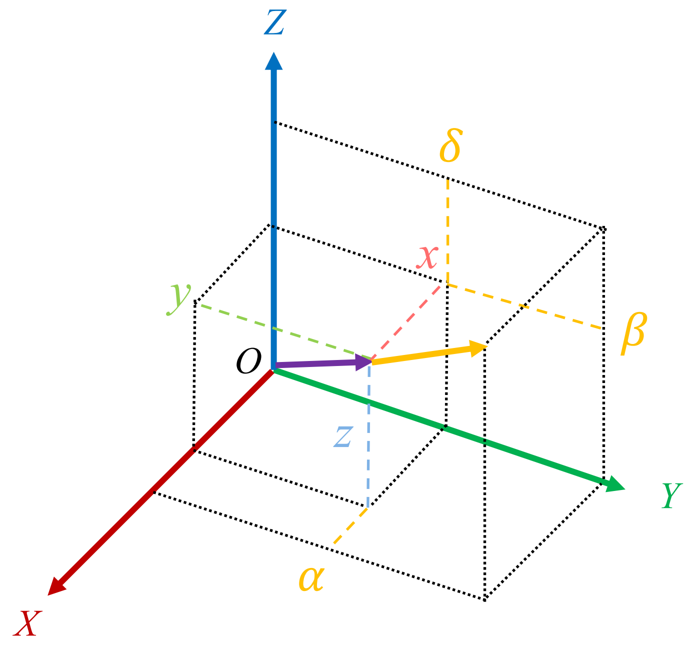
$$T(\alpha, \beta, \gamma)(P) = \begin{pmatrix} x + \alpha \\ y + \beta \\ z + \gamma \end{pmatrix}$$



Translation properties

- Let $P = (x, y, z)$ be a point, $\Theta = (\alpha, \beta, \gamma)$ be a vector and $T(\alpha, \beta, \gamma)$ be a translation:

$$T(\alpha,\beta,\gamma)(P) = \begin{pmatrix} x+\alpha \\ y+\beta \\ z+\gamma \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix} = \overrightarrow{OP} + \Theta$$



Translation properties

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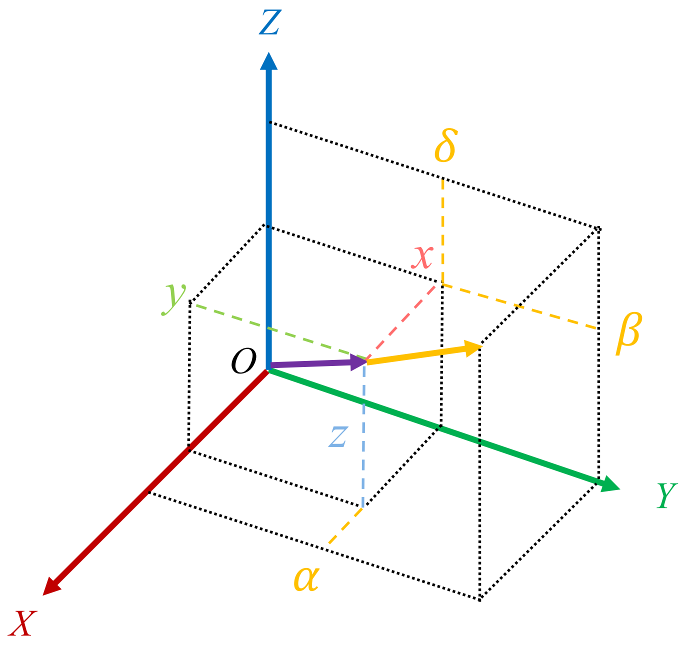
$$= \overrightarrow{OP} + T(\alpha,\beta,\gamma)(O)$$



Translation properties

- Let $P = (x, y, z)$ be a point, $\Theta = (\alpha, \beta, \gamma)$ be a vector and $T(\alpha, \beta, \gamma)$ be a translation:

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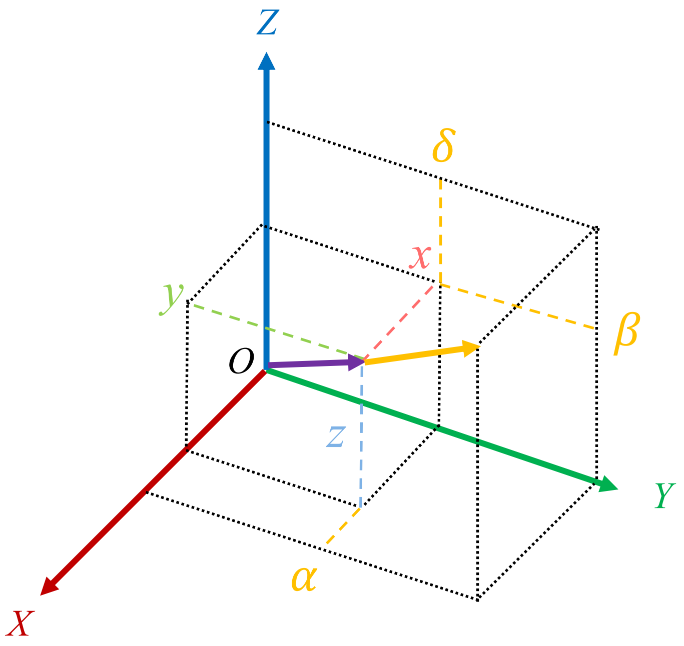
$$= \vec{\varphi}\left(\overrightarrow{OP}\right) + T(\alpha,\beta,\gamma)(O)$$

$\vec{\varphi}$ is the identity application: $\vec{\varphi}(X) = X$ (linear)



Translation properties

- Let $P = (x, y, z)$ be a point, $\Theta = (\alpha, \beta, \gamma)$ be a vector and $T(\alpha, \beta, \gamma)$ be a translation:

$$T(\alpha, \beta, \gamma)(P) = \vec{\varphi}\left(\overrightarrow{OP}\right) + T(\alpha, \beta, \gamma)(O)$$

- As $\vec{\varphi}$ is linear, $T(\alpha, \beta, \gamma)$ is an affine application that preserve:
  - Parallelism
  - Lengths
  - Angles

Translation properties

- Let $P = (x, y, z)$ be a point and $Q = (x_t, y_t, z_t)$ be the result of $T(\alpha, \beta, \gamma)(P)$:

$$Q = T(\alpha, \beta, \gamma)(P) = \begin{pmatrix} x + \alpha \\ y + \beta \\ z + \gamma \end{pmatrix} = \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix}$$

Translation properties

- Let $P = (x, y, z)$ be a point and $Q = (x_t, y_t, z_t)$ be the result of $T(\alpha, \beta, \gamma)(P)$:

$$Q = T(\alpha, \beta, \gamma)(P) = \begin{pmatrix} x + \alpha \\ y + \beta \\ z + \gamma \end{pmatrix} = \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix}$$

$$T(-\alpha, -\beta, -\gamma)(Q) = \begin{pmatrix} x_t - \alpha \\ y_t - \beta \\ z_t - \gamma \end{pmatrix}$$

Translation properties

- Let $P = (x, y, z)$ be a point and $Q = (x_t, y_t, z_t)$ be the result of $T(\alpha, \beta, \gamma)(P)$:

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$$T(-\alpha, -\beta, -\gamma)(Q) = \begin{pmatrix} x_t - \alpha \\ y_t - \beta \\ z_t - \gamma \end{pmatrix} = \begin{pmatrix} x + \alpha - \alpha \\ y + \beta - \beta \\ z + \gamma - \gamma \end{pmatrix}$$

Translation properties

- Let $P = (x, y, z)$ be a point and $Q = (x_t, y_t, z_t)$ be the result of $T(\alpha, \beta, \gamma)(P)$:

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$$T(-\alpha, -\beta, -\gamma)(Q) = \begin{pmatrix} x_t - \alpha \\ y_t - \beta \\ z_t - \gamma \end{pmatrix} = \begin{pmatrix} x + \alpha - \alpha \\ y + \beta - \beta \\ z + \gamma - \gamma \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = P$$

Translation properties

- Let $P = (x, y, z)$ be a point and $Q = (x_t, y_t, z_t)$ be the result of $T(\alpha, \beta, \gamma)(P)$:

$$Q = T(\alpha, \beta, \gamma)(P) = \begin{pmatrix} x + \alpha \\ y + \beta \\ z + \gamma \end{pmatrix} = \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix}$$

$$T(-\alpha, -\beta, -\gamma)(Q) = P$$

$$T(-\alpha, -\beta, -\gamma)\big(T(\alpha, \beta, \gamma)(P)\big) = P$$

Translation properties

- Let $P = (x, y, z)$ be a point and $Q = (x_t, y_t, z_t)$ be the result of $T(\alpha, \beta, \gamma)(P)$:

$$Q = T(\alpha, \beta, \gamma)(P) = \begin{pmatrix} x + \alpha \\ y + \beta \\ z + \gamma \end{pmatrix} = \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix}$$

$$T(-\alpha, -\beta, -\gamma)(Q) = P$$

$$\boldsymbol{T(-\alpha, -\beta, -\gamma)\big(T(\alpha, \beta, \gamma)(P)\big) = P}$$

- Translation $T(\alpha, \beta, \gamma)$ is invertible and its inverse is $T(-\alpha, -\beta, -\gamma)$

Translation

- Let $P = (x, y, z)$ a point and (α, β, γ) a vector. A translation, denoted $T(\alpha, \beta, \gamma)$, is an application such as:

$$T(\alpha,\beta,\gamma)(P) = \begin{pmatrix} x+\alpha \\ y+\beta \\ z+\gamma \end{pmatrix}$$

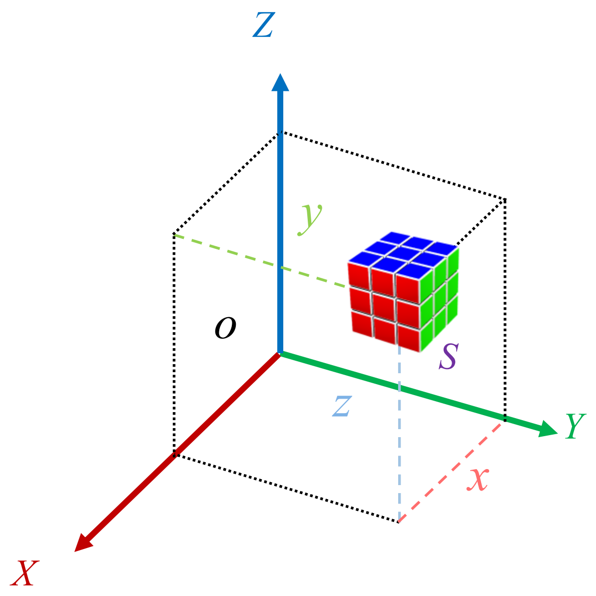
- Translation is an affine application

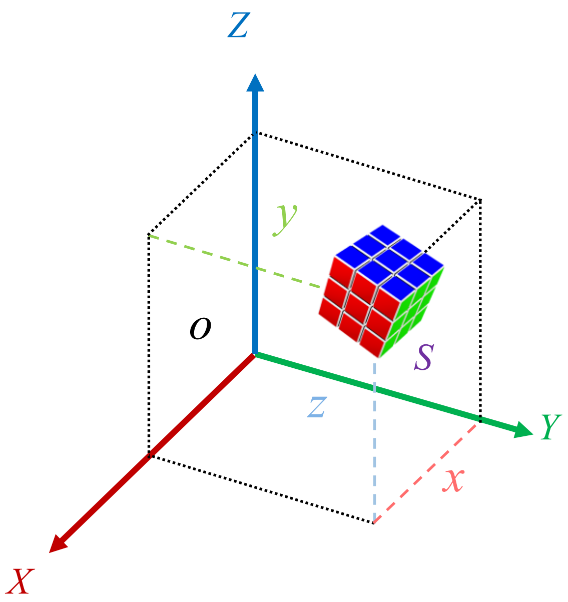
- Translation is invertible and its inverse is:

$$T^{-1}(\alpha,\beta,\gamma) = T(-\alpha,-\beta,-\gamma)$$

Representing 3D orientation

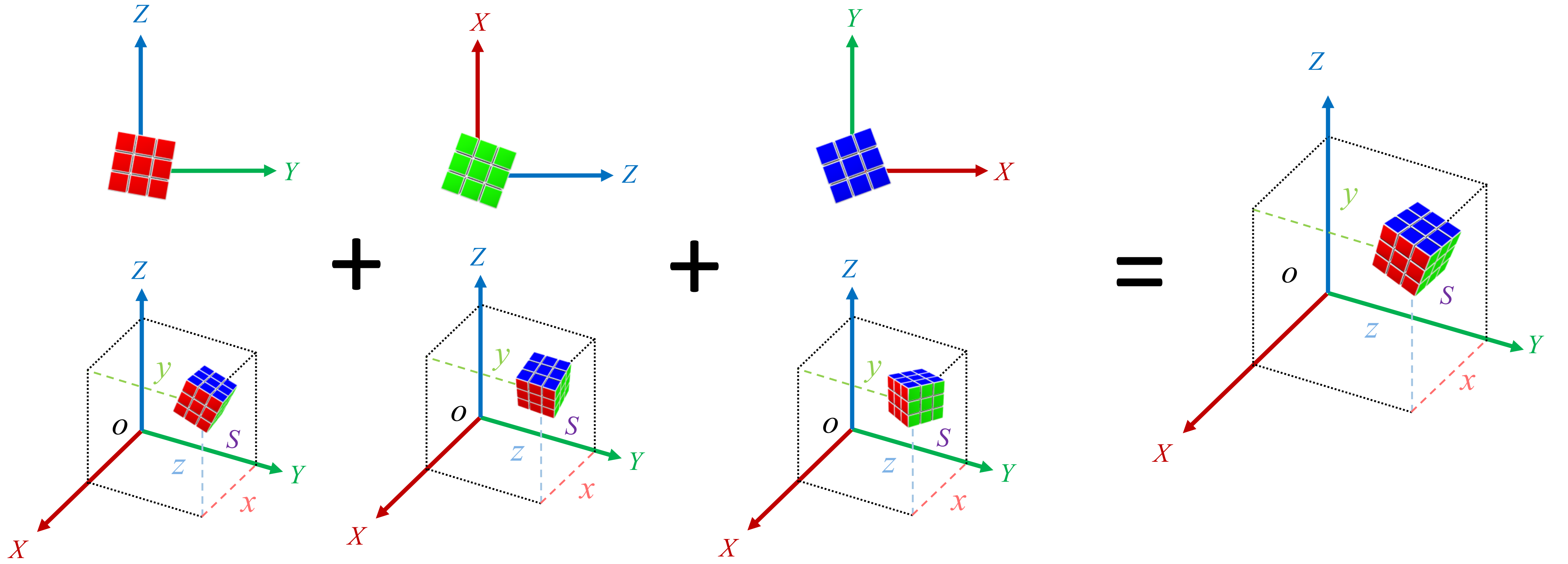
- 3D position is not enough to express location within 3D space
  - Two objects can share the **same position** but have **different orientation**





Representing 3D orientation

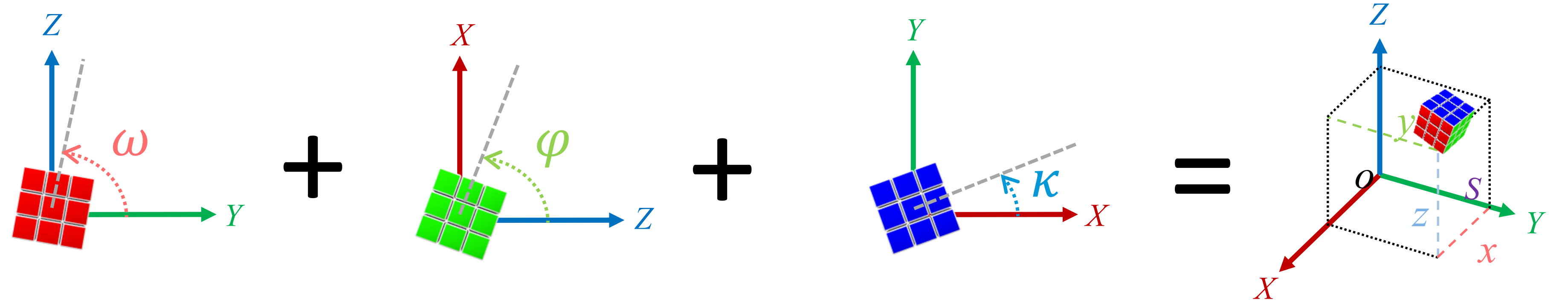
- Orientation of an object within a 3D space



3D orientation

*an **orientation** within a 3D space can be defined by three angles* $(\omega, \varphi, \kappa)$ *that describe the rotations around X, Y and Z axis, respectively.*

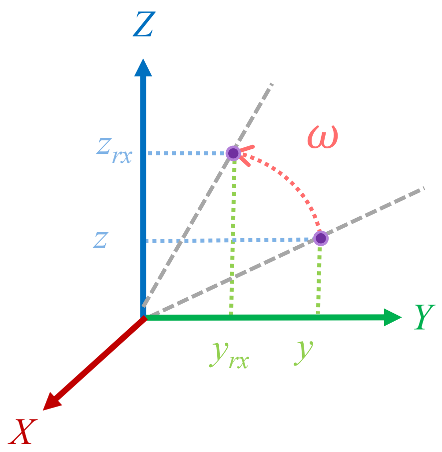
- Rotations around X, Y and Z axis are denoted $R_x(\omega)$, $R_y(\varphi)$ and $R_z(\kappa)$



$R_x(\omega)$ + $R_y(\varphi)$ + $R_z(\kappa)$ = $R_x(\omega)R_y(\varphi)R_z(\kappa)$

Axes rotation

- Rotation around X axis
  - Let $P = (x, y, z)$ be a point and ω be an angle. The result of the rotation of P around X axis by ω, denoted $R_x(\omega)$, is such as:

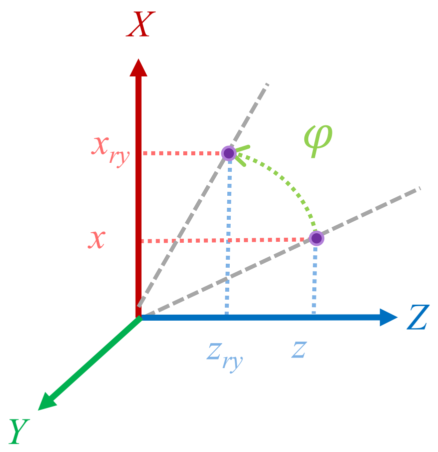


$$R_x(\omega)(P) = \begin{pmatrix} x \\ y\cos(\omega) - z\sin(\omega) \\ y\sin(\omega) + z\cos(\omega) \end{pmatrix}$$

α

Axes rotation

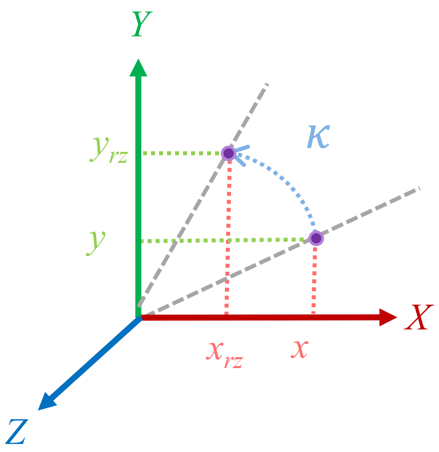
- Rotation around Y axis
  - Let $P = (x, y, z)$ be a point and φ be an angle. The result of the rotation of P around Y axis by φ, denoted $R_y(\varphi)$, is such as:



$$R_y(\varphi)(P) = \begin{pmatrix} x\cos(\varphi) + z\sin(\varphi) \\ y \\ z\cos(\varphi) - x\sin(\varphi) \end{pmatrix}$$

Axes rotation

- Rotation around Z axis
  - Let $P = (x, y, z)$ be a point and κ be an angle. The result of the rotation of P around Z axis by κ, denoted $R_y(\kappa)$, is such as:



$$R_z(\kappa)(P) = \begin{pmatrix} x\cos(\kappa) - y\sin(\kappa) \\ x\sin(\kappa) + y\cos(\kappa) \\ z \end{pmatrix}$$

Axes rotation properties

- Let $A = (x_a, y_a, z_a)$ and $B = (x_b, y_b, z_b)$ be two vectors

$$R_x(\omega)(\lambda A + \mu B) = \begin{pmatrix} \lambda x_a + \mu x_b \\ (\lambda y_a + \mu y_b)\cos(\omega) - (\lambda z_a + \mu z_b)\sin(\omega) \\ (\lambda y_a + \mu y_b)\, sin(\omega) + (\lambda z_a + \mu z_b)\cos(\omega) \end{pmatrix}$$

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Distribution

$$R_x(\omega)(\lambda A + \mu B) = \begin{pmatrix} \lambda x_a + \mu x_b \\ \lambda y_a \cos(\omega) + \mu y_b \cos(\omega) - \lambda z_a \sin(\omega) - \mu z_b \sin(\omega) \\ \lambda y_a \sin(\omega) + \mu y_b \sin(\omega) + \lambda z_a \cos(\omega) + \mu z_b \cos(\omega) \end{pmatrix}$$

Axes rotation properties

- Let $A = (x_a, y_a, z_a)$ and $B = (x_b, y_b, z_b)$ be two vectors

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$$R_x(\omega)(\lambda A + \mu B) = \begin{pmatrix} \lambda x_a + \mu x_b \\ \lambda y_a \cos(\omega) + \mu y_b \cos(\omega) - \lambda z_a \sin(\omega) - \mu z_b \sin(\omega) \\ \lambda y_a \sin(\omega) + \mu y_b \sin(\omega) + \lambda z_a \cos(\omega) + \mu z_b \cos(\omega) \end{pmatrix}$$

Factorization

$$R_x(\omega)(\lambda A + \mu B) = \begin{pmatrix} \lambda x_a + \mu x_b \\ \lambda(y_a \cos(\omega) - z_a \sin(\omega)) + \mu(y_b \cos(\omega) - z_b \sin(\omega)) \\ \lambda(y_a \sin(\omega) + z_a \cos(\omega)) + \mu(y_b \sin(\omega) + z_b \cos(\omega)) \end{pmatrix}$$

Axes rotation properties

- Let $A = (x_a, y_a, z_a)$ and $B = (x_b, y_b, z_b)$ be two vectors

$$R_x(\omega)(\lambda A + \mu B) = \begin{pmatrix} \lambda x_a + \mu x_b \\ \lambda(y_a \cos(\omega) - z_a \sin(\omega)) + \mu(y_b \cos(\omega) - z_b \sin(\omega)) \\ \lambda(y_a \sin(\omega) + z_a \cos(\omega)) + \mu(y_b \sin(\omega) + z_b \cos(\omega)) \end{pmatrix}$$

$$R_x(\omega)(\lambda A + \mu B) = \begin{pmatrix} \lambda x_a \\ \lambda(y_a \cos(\omega) - z_a \sin(\omega)) \\ \lambda(y_a \sin(\omega) + z_a \cos(\omega)) \end{pmatrix} + \begin{pmatrix} \mu x_b \\ \mu(y_b \cos(\omega) - z_b \sin(\omega)) \\ \mu(y_b \sin(\omega) + z_b \cos(\omega)) \end{pmatrix}$$

Vector decomposition

Axes rotation properties

- Let $A = (x_a, y_a, z_a)$ and $B = (x_b, y_b, z_b)$ be two vectors

$$R_x(\omega)(\lambda A + \mu B) = \begin{pmatrix} \lambda x_a + \mu x_b \\ \lambda(y_a\cos(\omega) - z_a\sin(\omega)) + \mu(y_b\cos(\omega) - z_b\sin(\omega)) \\ \lambda(y_a\sin(\omega) + z_a\cos(\omega)) + \mu(y_b\sin(\omega) + z_b\cos(\omega)) \end{pmatrix}$$

$$R_x(\omega)(\lambda A + \mu B) = \begin{pmatrix} \lambda x_a \\ \lambda(y_a\cos(\omega) - z_a\sin(\omega)) \\ \lambda(y_a\sin(\omega) + z_a\cos(\omega)) \end{pmatrix} + \begin{pmatrix} \mu x_b \\ \mu(y_b\cos(\omega) - z_b\sin(\omega)) \\ \mu(y_b\sin(\omega) + z_b\cos(\omega)) \end{pmatrix}$$

Factorization

$$R_x(\omega)(\lambda A + \mu B) = \lambda \begin{pmatrix} x_a \\ y_a\cos(\omega) - z_a\sin(\omega) \\ y_a\sin(\omega) + z_a\cos(\omega) \end{pmatrix} + \mu \begin{pmatrix} x_b \\ y_b\cos(\omega) - z_b\sin(\omega) \\ y_b\sin(\omega) + z_b\cos(\omega) \end{pmatrix}$$

Axes rotation properties

- Let $A = (x_a, y_a, z_a)$ and $B = (x_b, y_b, z_b)$ be two vectors

$$R_x(\omega)(\lambda A + \mu B) = \underbrace{\lambda \begin{pmatrix} x_a \\ y_a \cos(\omega) - z_a \sin(\omega) \\ y_a \sin(\omega) + z_a \cos(\omega) \end{pmatrix}}_{R_x(\omega)(A)} + \underbrace{\mu \begin{pmatrix} x_b \\ y_b \cos(\omega) - z_b \sin(\omega) \\ y_b \sin(\omega) + z_b \cos(\omega) \end{pmatrix}}_{R_x(\omega)(B)}$$

Axes rotation properties

- Let $A = (x_a, y_a, z_a)$ and $B = (x_b, y_b, z_b)$ be two vectors

$$R_x(\omega)(\lambda A + \mu B) = \lambda \underbrace{\begin{pmatrix} x_a \\ y_a \cos(\omega) - z_a \sin(\omega) \\ y_a \sin(\omega) + z_a \cos(\omega) \end{pmatrix}}_{R_x(\omega)(A)} + \mu \underbrace{\begin{pmatrix} x_b \\ y_b \cos(\omega) - z_b \sin(\omega) \\ y_b \sin(\omega) + z_b \cos(\omega) \end{pmatrix}}_{R_x(\omega)(B)}$$

⇓

$$R_x(\omega)(\lambda A + \mu B) = \lambda R_x(\omega)(A) + \mu R_x(\omega)(B)$$

The rotation $R_x(\omega)$ is a linear application

Axes rotation properties

- Exercise

Demonstrate the linearity of $R_y(\omega)$ and $R_z(\kappa)$

Axes rotation properties

- Let $A = (x_a, y_a, z_a)$ and $B = (x_b, y_b, z_b)$ such as

$$R_x(\omega)(A) = \begin{pmatrix} x_a \\ y_a \cos(\omega) - z_a \sin(\omega) \\ y_a \sin(\omega) + z_a \cos(\omega) \end{pmatrix} = \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = B$$

Axes rotation properties

- Let $A = (x_a, y_a, z_a)$ and $B = (x_b, y_b, z_b)$ such as

$$R_x(\omega)(A) = \begin{pmatrix} x_a \\ y_a \cos(\omega) - z_a \sin(\omega) \\ y_a \sin(\omega) + z_a \cos(\omega) \end{pmatrix} = \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = B$$

Applying rotation of $-\omega$ to B

$$R_x(-\omega)(B) = \begin{pmatrix} x_b \\ y_b \cos(-\omega) - z_b \sin(-\omega) \\ y_b \sin(-\omega) + z_b \cos(-\omega) \end{pmatrix}$$

Axes rotation properties

- Let $A = (x_a, y_a, z_a)$ and $B = (x_b, y_b, z_b)$ such as

$$R_x(\omega)(A) = \begin{pmatrix} x_a \\ y_a\cos(\omega) - z_a\sin(\omega) \\ y_a\sin(\omega) + z_a\cos(\omega) \end{pmatrix} = \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = B$$

$$R_x(-\omega)(B) = \begin{pmatrix} x_b \\ y_b\cos(-\omega) - z_b\sin(-\omega) \\ y_b\sin(-\omega) + z_b\cos(-\omega) \end{pmatrix}$$

Replacing

$$R_x(-\omega)(B) = \begin{pmatrix} x_a \\ (y_a\cos(\omega) - z_a\sin(\omega))\cos(-\omega) - (y_a\sin(\omega) + z_a\cos(\omega))\sin(-\omega) \\ (y_a\cos(\omega) - z_a\sin(\omega))\sin(-\omega) + (y_a\sin(\omega) + z_a\cos(\omega))\cos(-\omega) \end{pmatrix}$$

Axes rotation properties

- Let $A = (x_a, y_a, z_a)$ and $B = (x_b, y_b, z_b)$ such as

$$R_x(-\omega)(B) = \begin{pmatrix} x_a \\ (y_a\cos(\omega) - z_a\sin(\omega))\cos(-\omega) - (y_a\sin(\omega) + z_a\cos(\omega))\sin(-\omega) \\ (y_a\cos(\omega) - z_a\sin(\omega))\sin(-\omega) + (y_a\sin(\omega) + z_a\cos(\omega))\cos(-\omega) \end{pmatrix}$$

Axes rotation properties

- Let $A = (x_a, y_a, z_a)$ and $B = (x_b, y_b, z_b)$ such as

$$R_x(-\omega)(B) = \begin{pmatrix} x_a \\ (y_a\cos(\omega) - z_a\sin(\omega))\cos(-\omega) - (y_a\sin(\omega) + z_a\cos(\omega))\sin(-\omega) \\ (y_a\cos(\omega) - z_a\sin(\omega))\sin(-\omega) + (y_a\sin(\omega) + z_a\cos(\omega))\cos(-\omega) \end{pmatrix}$$

Distribution

$$R_x(-\omega)(B) = \begin{pmatrix} x_a \\ y_a\cos(\omega)\cos(-\omega) - z_a\sin(\omega)\cos(-\omega) - y_a\sin(\omega)\sin(-\omega) - z_a\cos(\omega)\sin(-\omega) \\ y_a\cos(\omega)\sin(-\omega) - z_a\sin(\omega)\sin(-\omega) + y_a\sin(\omega)\cos(-\omega) + z_a\cos(\omega)\cos(-\omega) \end{pmatrix}$$

Axes rotation properties

- Let $A = (x_a, y_a, z_a)$ and $B = (x_b, y_b, z_b)$ such as

$$R_x(-\omega)(B) = \begin{pmatrix} x_a \\ (y_a \cos(\omega) - z_a \sin(\omega)) \cos(-\omega) - (y_a \sin(\omega) + z_a \cos(\omega)) \sin(-\omega) \\ (y_a \cos(\omega) - z_a \sin(\omega)) \sin(-\omega) + (y_a \sin(\omega) + z_a \cos(\omega)) \cos(-\omega) \end{pmatrix}$$

$$R_x(-\omega)(B) = \begin{pmatrix} x_a \\ y_a \cos(\omega) \cos(-\omega) - z_a \sin(\omega) \cos(-\omega) - y_a \sin(\omega) \sin(-\omega) - z_a \cos(\omega) \sin(-\omega) \\ y_a \cos(\omega) \sin(-\omega) - z_a \sin(\omega) \sin(-\omega) + y_a \sin(\omega) \cos(-\omega) + z_a \cos(\omega) \cos(-\omega) \end{pmatrix}$$

$$R_x(-\omega)(B) = \begin{pmatrix} x_a \\ y_a \cos(\omega) \cos(-\omega) - y_a \sin(\omega) \sin(-\omega) - z_a \sin(\omega) \cos(-\omega) - z_a \cos(\omega) \sin(-\omega) \\ y_a \cos(\omega) \sin(-\omega) + y_a \sin(\omega) \cos(-\omega) + z_a \cos(\omega) \cos(-\omega) - z_a \sin(\omega) \sin(-\omega) \end{pmatrix}$$

Grouping

Axes rotation properties

- Let $A = (x_a, y_a, z_a)$ and $B = (x_b, y_b, z_b)$ such as

$$R_x(-\omega)(B) = \begin{pmatrix} x_a \\ y_a \cos(\omega)\cos(-\omega) - y_a \sin(\omega)\sin(-\omega) - z_a \sin(\omega)\cos(-\omega) - z_a \cos(\omega)\sin(-\omega) \\ y_a \cos(\omega)\sin(-\omega) + y_a \sin(\omega)\cos(-\omega) + z_a \cos(\omega)\cos(-\omega) - z_a \sin(\omega)\sin(-\omega) \end{pmatrix}$$

Axes rotation properties

- Let $A = (x_a, y_a, z_a)$ and $B = (x_b, y_b, z_b)$ such as

$$R_x(-\omega)(B) = \begin{pmatrix} x_a \\ y_a \cos(\omega)\cos(-\omega) - y_a \sin(\omega)\sin(-\omega) - z_a \sin(\omega)\cos(-\omega) - z_a \cos(\omega)\sin(-\omega) \\ y_a \cos(\omega)\sin(-\omega) + y_a \sin(\omega)\cos(-\omega) + z_a \cos(\omega)\cos(-\omega) - z_a \sin(\omega)\sin(-\omega) \end{pmatrix}$$

Factorization

$$R_x(-\omega)(B) = \begin{pmatrix} x_a \\ y_a(\cos(\omega)\cos(-\omega) - \sin(\omega)\sin(-\omega)) - z_a(\sin(\omega)\cos(-\omega) + \cos(\omega)\sin(-\omega)) \\ y_a(\cos(\omega)\sin(-\omega) + \sin(\omega)\cos(-\omega)) + z_a(\cos(\omega)\cos(-\omega) - \sin(\omega)\sin(-\omega)) \end{pmatrix}$$

Axes rotation properties

- Let $A = (x_a, y_a, z_a)$ and $B = (x_b, y_b, z_b)$ such as

$$R_x(-\omega)(B) = \begin{pmatrix} x_a \\ y_a\cos(\omega)\cos(-\omega) - y_a\sin(\omega)\sin(-\omega) - z_a\sin(\omega)\cos(-\omega) - z_a\cos(\omega)\sin(-\omega) \\ y_a\cos(\omega)\sin(-\omega) + y_a\sin(\omega)\cos(-\omega) + z_a\cos(\omega)\cos(-\omega) - z_a\sin(\omega)\sin(-\omega) \end{pmatrix}$$

$$R_x(-\omega)(B) = \begin{pmatrix} x_a \\ y_a(\cos(\omega)\cos(-\omega) - \sin(\omega)\sin(-\omega)) - z_a(\sin(\omega)\cos(-\omega) + \cos(\omega)\sin(-\omega)) \\ y_a(\cos(\omega)\sin(-\omega) + \sin(\omega)\cos(-\omega)) + z_a(\cos(\omega)\cos(-\omega) - \sin(\omega)\sin(-\omega)) \end{pmatrix}$$

$$\cos a\cos b - \sin a\sin b = \cos(a+b)$$

$$R_x(-\omega)(B) = \begin{pmatrix} x_a \\ y_a\cos(\omega-\omega) - z_a(\sin(\omega)\cos(-\omega) + \cos(\omega)\sin(-\omega)) \\ y_a(\cos(\omega)\sin(-\omega) + \sin(\omega)\cos(-\omega)) + z_a\cos(\omega-\omega) \end{pmatrix}$$

Axes rotation properties

- Let $A = (x_a, y_a, z_a)$ and $B = (x_b, y_b, z_b)$ such as

$$R_x(-\omega)(B) = \begin{pmatrix} x_a \\ y_a \cos(\omega - \omega) - z_a(\sin(\omega)\cos(-\omega) + \cos(\omega)\sin(-\omega)) \\ y_a(\cos(\omega)\sin(-\omega) + \sin(\omega)\cos(-\omega)) + z_a \cos(\omega - \omega) \end{pmatrix}$$

Axes rotation properties

- Let $A = (x_a, y_a, z_a)$ and $B = (x_b, y_b, z_b)$ such as

$$R_x(-\omega)(B) = \begin{pmatrix} x_a \\ y_a \cos(\omega - \omega) - z_a(\sin(\omega)\cos(-\omega) + \cos(\omega)\sin(-\omega)) \\ y_a(\cos(\omega)\sin(-\omega) + \sin(\omega)\cos(-\omega)) + z_a \cos(\omega - \omega) \end{pmatrix}$$

$$\sin a \cos b + \cos a \sin b = \sin(a + b)$$

$$R_x(-\omega)(B) = \begin{pmatrix} x_a \\ y_a \cos(\omega - \omega) - z_a \sin(\omega - \omega) \\ y_a \sin(\omega - \omega) + z_a \cos(\omega - \omega) \end{pmatrix}$$

Axes rotation properties

- Let $A = (x_a, y_a, z_a)$ and $B = (x_b, y_b, z_b)$ such as

$$R_x(-\omega)(B) = \begin{pmatrix} x_a \\ y_a \cos(\omega - \omega) - z_a(\sin(\omega)\cos(-\omega) + \cos(\omega)\sin(-\omega)) \\ y_a(\cos(\omega)\sin(-\omega) + \sin(\omega)\cos(-\omega)) + z_a \cos(\omega - \omega) \end{pmatrix}$$

$$R_x(-\omega)(B) = \begin{pmatrix} x_a \\ y_a \cos(\omega - \omega) - z_a \sin(\omega - \omega) \\ y_a \sin(\omega - \omega) + z_a \cos(\omega - \omega) \end{pmatrix}$$

$$\cos 0 = 1, \sin 0 = 0$$

$$R_x(-\omega)(B) = \begin{pmatrix} x_a \\ y_a \times 1 - z_a \times 0 \\ y_a \times 0 + z_a \times 1 \end{pmatrix} = \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix}$$

Axes rotation properties

- Let $A = (x_a, y_a, z_a)$ and $B = (x_b, y_b, z_b)$ such as

$$R_x(\omega)(A) = \begin{pmatrix} x_a \\ y_a\cos(\omega) - z_a\sin(\omega) \\ y_a\sin(\omega) + z_a\cos(\omega) \end{pmatrix} = \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = B \qquad R_x(-\omega)(B) = \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} = A$$

$$R_x(\omega)\big(R_x(-\omega)(B)\big) = R_x(\omega)(A) = B$$

$$R_x(\omega)\big(R_x(-\omega)(B)\big) = B$$

The rotation $R_x(\omega)$ is invertible and its inverse is $R_x(-\omega)$

Axes rotation properties

- Exercise

Demonstrate the invertibility of $R_y(\omega)$ and $R_z(\kappa)$

Axes rotation

- Let $P = (x, y, z\ )$ be a point and let ω, φ and κ be three angles

$$R_x(\omega)(P) = \begin{pmatrix} x \\ y\cos(\omega) - z\sin(\omega) \\ y\sin(\omega) + z\cos(\omega) \end{pmatrix} \quad R_y(\varphi)(P) = \begin{pmatrix} x\cos(\varphi) + z\sin(\varphi) \\ y \\ z\cos(\varphi) - x\sin(\varphi) \end{pmatrix} \quad R_z(\kappa)(P) = \begin{pmatrix} x\cos(\kappa) - y\sin(\kappa) \\ x\sin(\kappa) + y\cos(\kappa) \\ z \end{pmatrix}$$

- Rotation $R_x(\omega)$, $R_y(\varphi)$ and $R_z(\kappa)$ are linear applications

- Rotation $R_x(\omega)$, $R_y(\varphi)$ and $R_z(\kappa)$ are invertible and their inverse are $R_x^{-1}(\omega) = R_x(-\omega)$, $R_y^{-1}(\varphi) = R_x(-\varphi)$ and $R_z^{-1}(\kappa) = R_z(-\kappa)$

Combined rotation

- $R_x(\omega)R_y(\varphi)R_z(\kappa)$

$$x_{rx} = x$$
$$y_{rx} = y\cos(\omega) - z\sin(\omega)$$
$$z_{rx} = y\,sin(\omega) + z\cos(\omega)$$

$R_x(\omega)$

$$x_{rxy} = x_{rx}\cos(\varphi) + z_{rx}\sin(\varphi)$$
$$y_{rxy} = y_{rx}$$
$$z_{rxy} = z_{rx}\cos(\varphi) - x_{rx}\sin(\varphi)$$

$R_y(\varphi) \circ R_x(\omega)$

$$x_{rxyz} = x_{rxy}\cos(\kappa) - y_{rxy}\sin(\kappa)$$
$$y_{rxyz} = x_{rxy}\sin(\kappa) + y_{rxy}cos(\kappa)$$
$$z_{rxyz} = z_{rxy}$$

$R_z(\kappa) \circ R_y(\varphi) \circ R_x(\omega)$

- Functional notation

$$R_x(\omega)R_y(\varphi)R_z(\kappa) \Rightarrow R_z(\kappa) \circ R_y(\varphi) \circ R_x(\omega)$$

Rotation computation

- $R_x(\omega)R_y(\varphi)R_z(\kappa)$

$$x_{rxyz} = x_{rxy}\cos(\kappa) - y_{rxy}\sin(\kappa)$$

$$y_{rxyz} = x_{rxy}\sin(\kappa) + y_{rxy}cos(\kappa)$$

$$z_{rxyz} = z_{rxy}$$

$$x_{rxy} = x_{rx}\cos(\varphi) + z_{rx}\sin(\varphi)$$

$$y_{rxy} = y_{rx}$$

$$z_{rxy} = z_{rx}\cos(\varphi) - x_{rx}\sin(\varphi)$$

$$x_{rxyz} = (x_{rx}\cos(\varphi) + z_{rx}\sin(\varphi))\cos(\kappa) - y_{rx}\sin(\kappa)$$

$$y_{rxyz} = (x_{rx}\cos(\varphi) + z_{rx}\sin(\varphi))\sin(\kappa) + y_{rx}cos(\kappa)$$

$$z_{rxyz} = z_{rx}\cos(\varphi) - x_{rx}\sin(\varphi)$$

Rotation computation

- $R_x(\omega)R_y(\varphi)R_z(\kappa)$

$$x_{rx} = x$$

$$y_{rx} = y\cos(\omega) - z\sin(\omega)$$

$$z_{rx} = y\,sin(\omega) + z\cos(\omega)$$

$$x_{rxyz} = (x_{rx}\cos(\varphi) + z_{rx}\sin(\varphi))\cos(\kappa) - y_{rx}\sin(\kappa)$$

$$y_{rxyz} = (x_{rx}\cos(\varphi) + z_{rx}\sin(\varphi))\sin(\kappa) + y_{rx}cos(\kappa)$$

$$z_{rxyz} = z_{rx}\cos(\varphi) - x_{rx}\sin(\varphi)$$

$$x_{rxyz} = (x\cos(\varphi) + (y\,sin(\omega) + z\cos(\omega))\sin(\varphi))\cos(\kappa) - (y\cos(\omega) - z\sin(\omega))\sin(\kappa)$$

$$y_{rxyz} = (x\cos(\varphi) + (y\,sin(\omega) + z\cos(\omega))\sin(\varphi))\sin(\kappa) + (y\cos(\omega) - z\sin(\omega))cos(\kappa)$$

$$z_{rxyz} = (y\,sin(\omega) + z\cos(\omega))\cos(\varphi) - x\sin(\varphi)$$

Rotation computation

- $R_x(\omega)R_y(\varphi)R_z(\kappa)$

$$x_{rxyz} = (x\cos(\varphi) + (y\,sin(\omega) + z\cos(\omega))\sin(\varphi))\cos(\kappa) - (y\cos(\omega) - z\sin(\omega))\sin(\kappa)$$

$$y_{rxyz} = (x\cos(\varphi) + (y\,sin(\omega) + z\cos(\omega))\sin(\varphi))\sin(\kappa) + (y\cos(\omega) - z\sin(\omega))cos(\kappa)$$

$$z_{rxyz} = (y\,sin(\omega) + z\cos(\omega))\cos(\varphi) - x\sin(\varphi)$$

$$x_{rxyz} = x\cos(\varphi)\cos(\kappa) + y\,sin(\omega)\sin(\varphi)\cos(\kappa) + z\cos(\omega)\sin(\varphi)\cos(\kappa) - y\,cos(\omega)\sin(\kappa) + z\sin(\omega)\sin(\kappa)$$

$$y_{rxyz} = x\cos(\varphi)\sin(\kappa) + y\,sin(\omega)\sin(\varphi)\sin(\kappa) + z\cos(\omega)\sin(\varphi)\sin(\kappa) + y\cos(\omega)cos(\kappa) - z\sin(\omega)\,cos(\kappa)$$

$$z_{rxyz} = (y\,sin(\omega)\cos(\varphi) + z\cos(\omega)\cos(\varphi)) - x\sin(\varphi)$$

Combined rotation

- $R_x(\omega)R_y(\varphi)R_z(\kappa)$

$$R_x(\omega)R_y(\varphi)R_z(\kappa) = \begin{pmatrix} x\cos(\varphi)\cos(\kappa) + y(\sin(\omega)\sin(\varphi)\cos(\kappa) - \cos(\omega)\sin(\kappa)) + z(\cos(\omega)\sin(\varphi)\cos(\kappa) + \sin(\omega)\sin(\kappa)) \\ x\cos(\varphi)\sin(\kappa) + y(\sin(\omega)\sin(\varphi)\sin(\kappa) + \cos(\omega)\cos(\kappa)) + z(\cos(\omega)\sin(\varphi)\sin(\kappa) - \sin(\omega)\cos(\kappa)) \\ -x\sin(\varphi) + y\sin(\omega)\cos(\varphi) + z\cos(\omega)\cos(\varphi) \end{pmatrix}$$

- Rotation $R_z(\kappa) \circ R_y(\varphi) \circ R_x(\omega)$ is a linear application as it combine three linear applications
- Rotation $R_z(\kappa) \circ R_y(\varphi) \circ R_x(\omega)$ is invertible as it combine three invertible applications and its inverse is $R_x(-\omega) \circ R_y(-\varphi) \circ R_z(-\kappa)$

3D Location

a location within a 3D space is defined by a position (x, y, z) *and an orientation* $(\omega, \varphi, \kappa)$.

Translation

$T(\alpha, \beta, \delta)$

Rotation

$R_x(\omega)R_y(\varphi)R_z(\kappa)$

a location within a 3D space is defined by a translation $T(\alpha, \beta, \delta)$ *and a composed rotation* $R_x(\omega)R_y(\varphi)R_z(\kappa)$

Rigid-body Transformation

*A transformation F that is obtained by the composition of rotations and translations is called **Rigid-body Transformation***

- Rigid transformation preserve:
  - Parallelism
  - Length

3D Transforms

Transform	Computation	Linearity	Distances	Angles
Translation	$(\alpha + x, \beta + y, \gamma + z)$	Affine	✓	✓
Rotation	$\begin{pmatrix} x \\ y\cos(\omega) - z\sin(\omega) \\ y\sin(\omega) + z\cos(\omega) \end{pmatrix}$ $\begin{pmatrix} x\cos(\varphi) + z\sin(\varphi) \\ y \\ z\cos(\varphi) - x\sin(\varphi) \end{pmatrix}$ $\begin{pmatrix} x\cos(\kappa) - y\sin(\kappa) \\ x\sin(\kappa) + y\cos(\kappa) \\ z \end{pmatrix}$	Linear	✓	✓
Scale (uniform)	(sx, sy, sz)	Linear	✗	✓
Scale	$(s_x x, s_y y, s_z z)$	Linear	✗	✗

Let $u_s = S_{s_x,s_y,s_z}(u) = \left(s_x u_x, s_y u_y, s_z u_z\right)$ and $v_s = S_{s_x,s_y,s_z}(v) = \left(s_x v_x, s_y v_y, s_z v_z\right)$

$$u_s \cdot v_s = s_x u_x \times s_x v_x + s_y u_y \times s_y v_y + s_z u_z \times s_z v_z = {s_x}^2 u_x v_x + {s_y}^2 u_y v_y + {s_z}^2 u_z v_z$$

$$u_s \cdot v_s = s^2 u_x v_x + s^2 u_y v_y + s^2 u_z v_z$$

$$u_s \cdot v_s = s^2\left(u_x v_x + u_y v_y + u_z v_z\right) = s^2(u \cdot v)$$

$$\cos\theta_s = \frac{u_s \cdot v_s}{\|u_s\|\|v_s\|} = \frac{s^2(u\cdot v)}{\sqrt{s^2(u\cdot u)}\sqrt{s^2(v\cdot v)}} = \frac{s^2(u\cdot v)}{s\sqrt{(u\cdot u)}\, s\sqrt{(v\cdot v)}} = \frac{s^2(u\cdot v)}{s^2\sqrt{(u\cdot u)}\sqrt{(v\cdot v)}} = \frac{u\cdot v}{\sqrt{(u\cdot u)}\sqrt{(v\cdot v)}} = \frac{u\cdot v}{\|u\|\|v\|} = \cos\theta$$