# Computer Vision

# Homogeneous Coordinates

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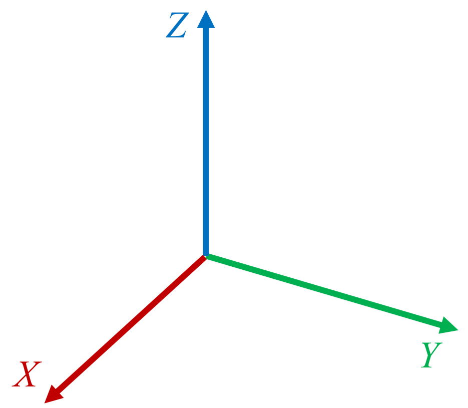
## Representing 3D space

3D space can be represented with Euclidean Space

- Vector space on $\mathbb{R}^3$
- Orthonormal basis of 3 vectors denoted X, Y and Z
- dot product $\cdot$ and cross product $\times$



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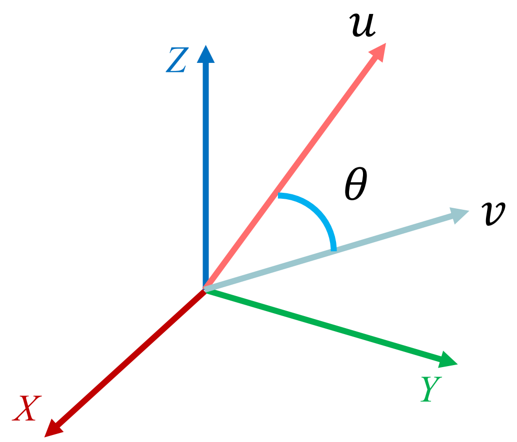
Mathematical properties

- Euclidean norm $\|u\| = \sqrt{u \cdot u}$
- Angle: $\cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$



# Representing Transforms

| Transform | Computation | Linearity | Distances | Angles |
|---|---|---|---|---|
| Translation | $(\alpha + x, \beta + y, \gamma + z)$ | Affine | ✓ | ✓ |
| Rotation | $\begin{pmatrix} x \\ y\cos(\omega) - z\sin(\omega) \\ y\sin(\omega) + z\cos(\omega) \end{pmatrix}$ $\begin{pmatrix} x\cos(\varphi) + z\sin(\varphi) \\ y \\ z\cos(\varphi) - x\sin(\varphi) \end{pmatrix}$ $\begin{pmatrix} x\cos(\kappa) - y\sin(\kappa) \\ x\sin(\kappa) + y\cos(\kappa) \\ z \end{pmatrix}$ | Linear | ✓ | ✓ |
| Scale (uniform) | $(sx, sy, sz)$ | Linear | ✗ | ✓ |
| Scale | $(s_x x, s_y y, s_z z)$ | Linear | ✗ | ✗ |

## Representing Transforms

Complex computation

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# Representing Transforms

Various properties

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## Representing 3D space

- Limits
  - No uniform representation of transformations
  - Computation can be complex
- Needs
  - A consistent mathematical formalism
  - Good computational properties

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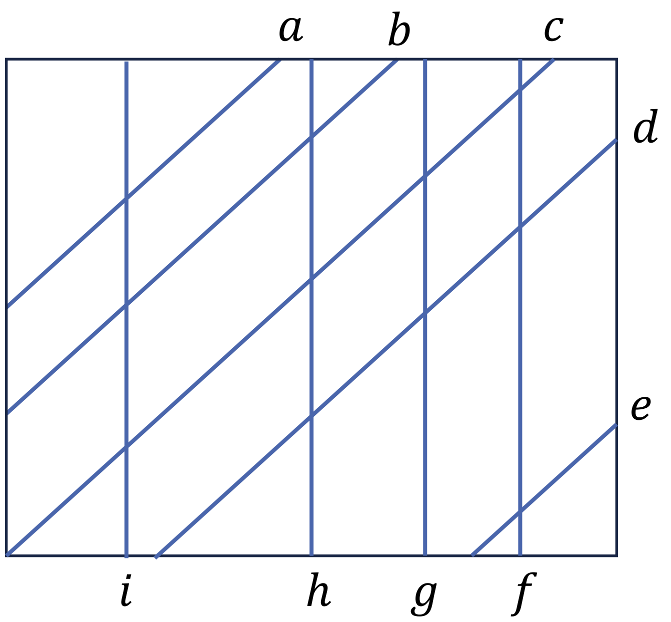
## Definition[Equivalence class]

*Let $E$ be a set with an equivalence relation $\sim$ and let $x \in E$. The equivalence class of $x$ for $\sim$ is the subset, denoted $\bar{x}^{\sim}$ such as:*

$$\bar{x}^{\sim} = \{y \in E, x \sim y\}$$

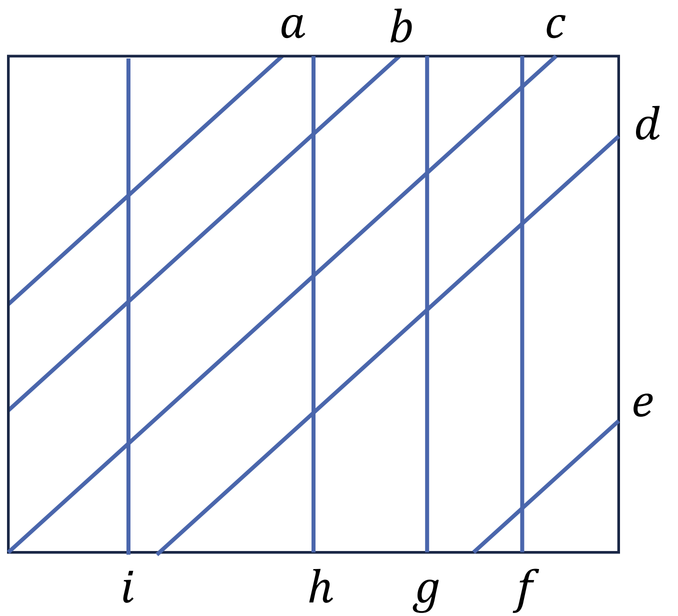
# Example[Equivalence]

Let $E = \{a, b, c, d, e, f, g, h, i\}$ be a set of lines within the plane



# Example[Equivalence]

Let $E = \{a, b, c, d, e, f, g, h, i\}$ be a set of lines within the plane and let $\sim$ a binary operation defined such as $\forall x, y \in E, x \sim y \rightarrow x\ parallel\ to\ y$



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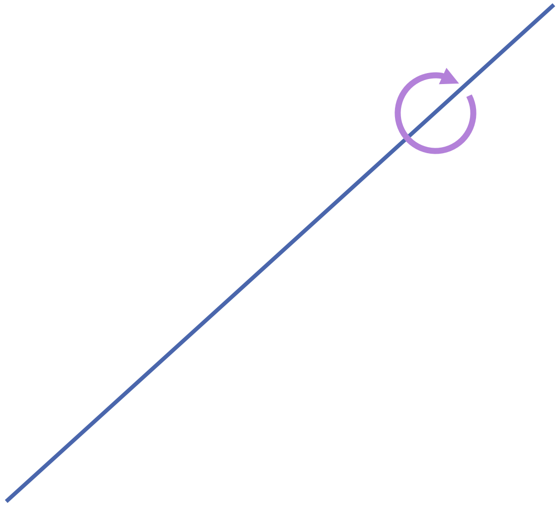
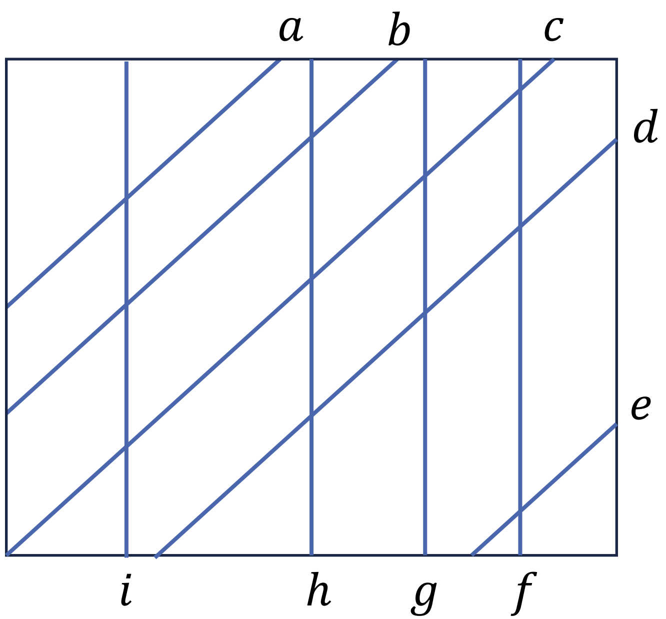
  A line $x$ is parallel to itself





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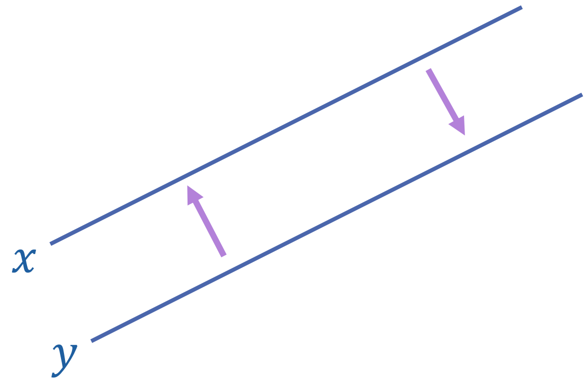


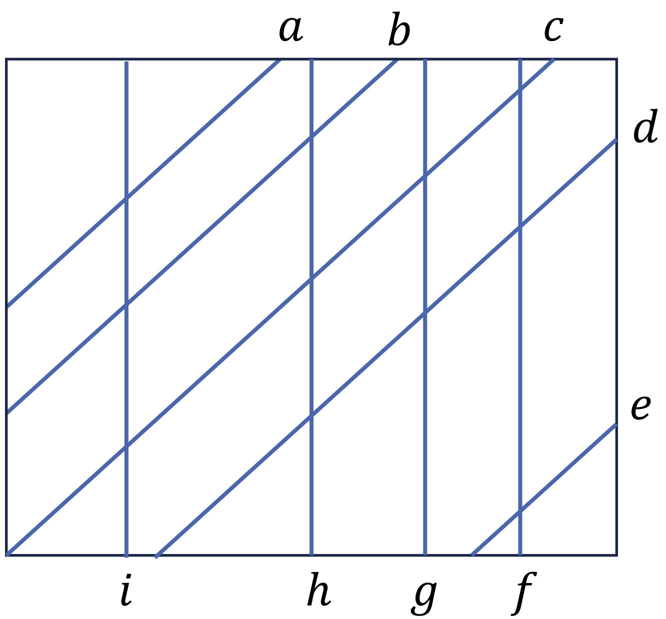

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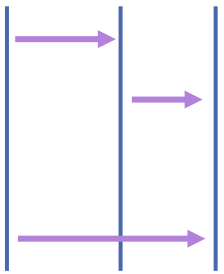

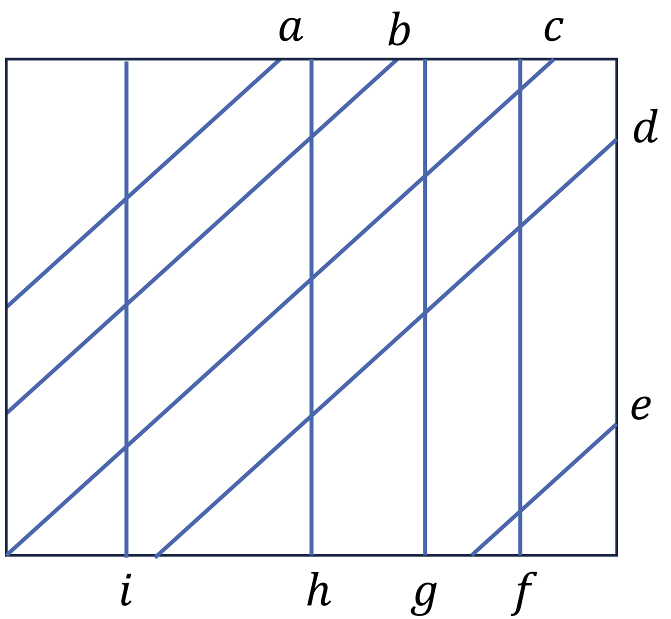

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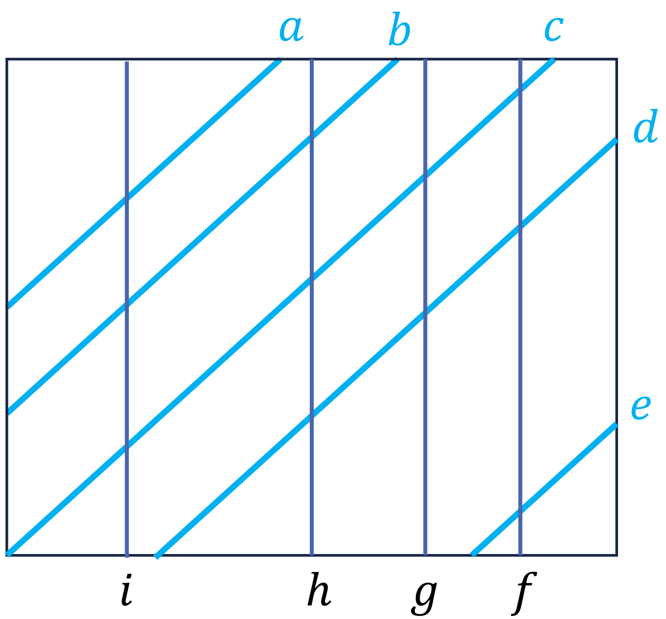
- The subset $G = \{a, b, c, d, e\}$ is an equivalent class



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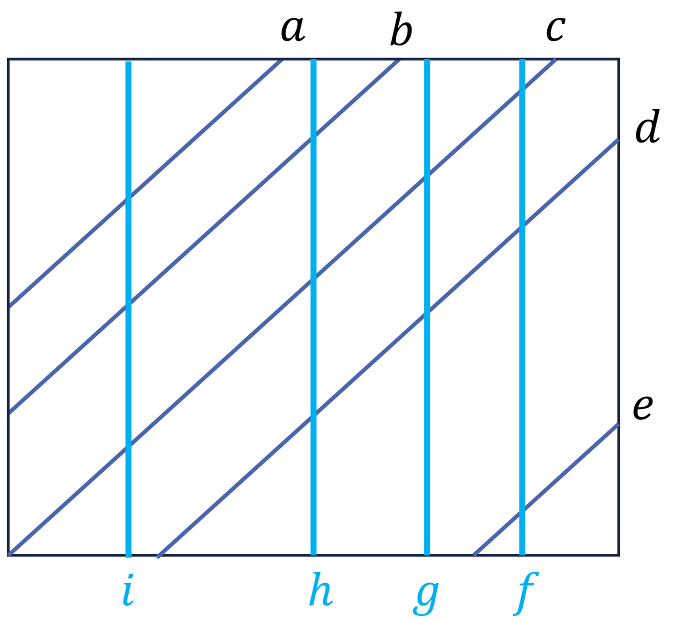

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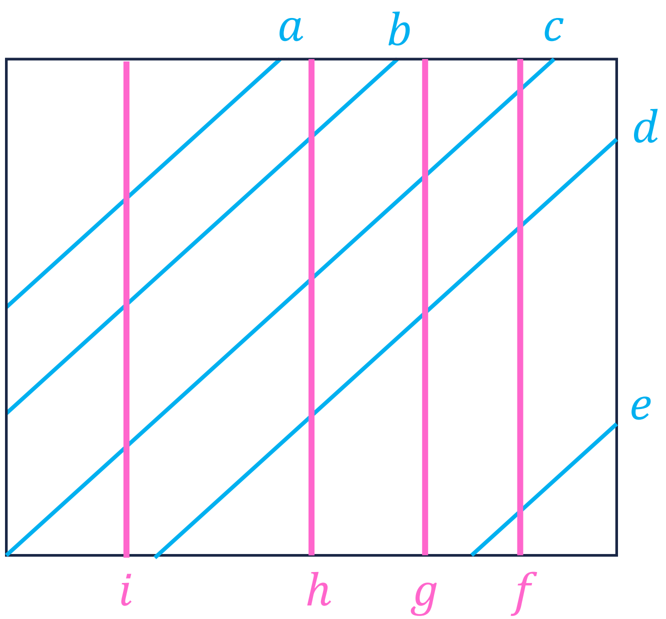

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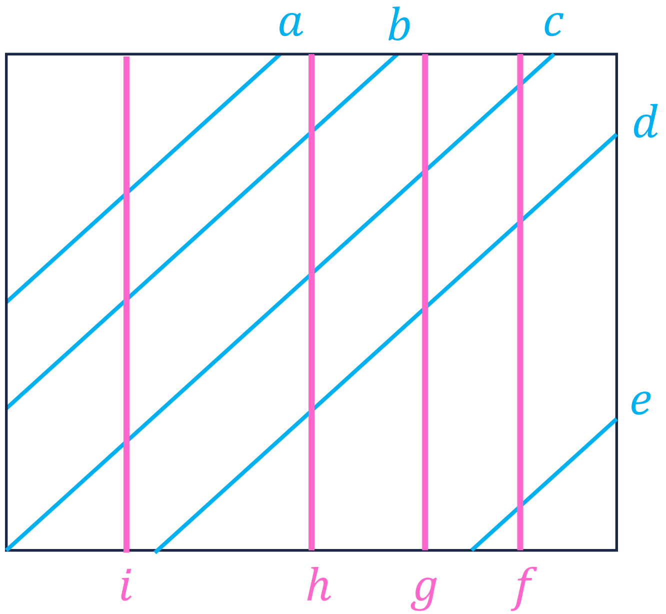

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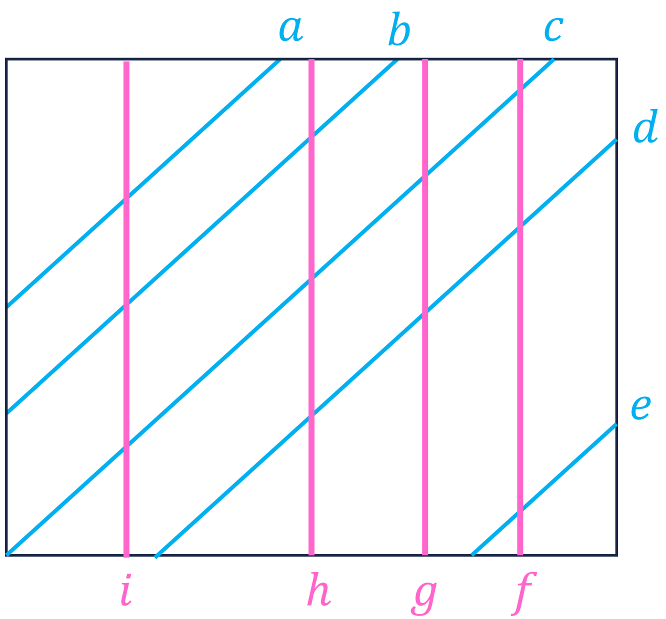
Equivalent classes form a partition



## Definition[Projective space]

*Let* $E$ *be a vector space on a field* $\mathbb{K}$*. The* ***projective space*** $\boldsymbol{P(E)}$ *is the set of equivalence classes of* $E \setminus \{0_E\}$ *under an equivalence relation, denoted* $\sim$*, such as:*

$$\forall \mathcal{X}, \mathcal{Y} \in E \setminus \{0_E\},\ \mathcal{X} \sim \mathcal{Y} \text{ if } \exists \lambda \in \mathbb{K},\ \lambda \neq 0, \mathcal{X} = \lambda \mathcal{Y}$$

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## Property[Projective space dimension]

Let $E^n$ be a vector space of finite dimension $n$. The projective space $P(E^n)$ has a dimension of $n - 1$.

# Definition[Homogeneous vector]

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## Interpretation

- A homogeneous vector describes an equivalence class of collinear vectors
- A homogenous vector has an infinity of representations:

$$\underbrace{\chi = (1, 4, 6)}_{\lambda = 1} = \underbrace{\left(\frac{1}{2}, 2,3\right)}_{\lambda = 1/2} = \underbrace{(12,48,72)}_{\lambda = 12} = \cdots = (\lambda \times 1, \lambda \times 4, \lambda \times 6)$$

# Homogeneous Representation

- **Definition**: A homogeneous vector $\mathcal{X}$ of dimension $n$ is a tuple made of $n$ **homogenous coordinates**:

$$\forall \lambda \in \mathbb{R}^* \quad \mathcal{X} = \lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$$

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- A homogeneous vector $\mathcal{X} = \lambda(x_1, \dots, x_n)$ can be represented by a column matrix, denoted $\mathrm{X}$, such as:

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## Euclidean to homogeneous Representation

- **Definition:** Let $a = (a_1, \dots, a_n)$ be a vector within a Euclidean space of dimension $n$. A **homogeneous representation** of $a$ is a vector of dimension $n+1$, denoted $\mathcal{A}$, such as:

$$\mathcal{A} = (\lambda a_1, \dots, \lambda a_n, \lambda) \equiv A = \begin{bmatrix} \lambda a_1 \\ \cdots \\ \lambda a_n \\ \lambda \end{bmatrix} = \begin{bmatrix} \lambda a_1 & \dots & \lambda a_n & \lambda \end{bmatrix}^{\top}$$

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- For the sake of simplicity, $\lambda$ is usually set to $1$

# Euclidean to homogeneous Representation

- **Example:** From Euclidean to homogeneous

$$(1, 4, 3)$$

Euclidean vector

# Euclidean to homogeneous Representation

- **Example:** From Euclidean to homogeneous

$$\underbrace{(1,4,3)}_{\text{Euclidean vector}} \rightarrow \underbrace{[1,4,3,1]^\top = 2[1,4,3,1]^\top = \cdots = \lambda[1,4,3,1]^\top}_{\text{Homogeneous representations}}$$

# Euclidean to homogeneous Representation

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- **Warning:** Homogeneous representation of the Euclidean space origin **is not** the origin of homogeneous space:

$$(0,0,0) \rightarrow [0,0,0,1]^\top = 2[0,0,0,1]^\top = \cdots = \lambda[0,0,0,1]^\top$$

## Homogeneous to Euclidean representation

- Let $\mathcal{A} = [a_1 \quad \dots \quad a_n \quad w]^\top$ be a vector within a homogeneous space of dimension $n + 1$, with $w \neq 0$.

  The Euclidean representation of $\mathcal{A}$ is a vector of dimension $n$, denoted $A$ such as:

$$A = \left(\frac{a_1}{w}, \dots, \frac{a_n}{w}\right)$$

# Homogeneous to Euclidean representation

- **Example:** From homogeneous to Euclidean

$$[2 \quad 8 \quad 6 \quad 2]^T$$

homogeneous vector

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$$\underbrace{[2 \quad 8 \quad 6 \quad 2]^{\mathrm{T}}}_{\text{homogeneous vector}} \rightarrow \underbrace{\left(\frac{2}{2}, \frac{8}{2}, \frac{6}{2}\right) = (1,4,3)}_{\text{Euclidean representation}}$$

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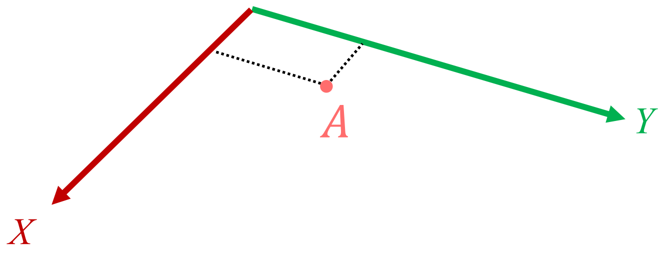
- **Warning:** Homogeneous space origin **cannot be represented** within Euclidean space:

$$[0,0,0,0]^\mathsf{T} \rightarrow \left(\frac{0}{0},\frac{0}{0},\frac{0}{0}\right)$$

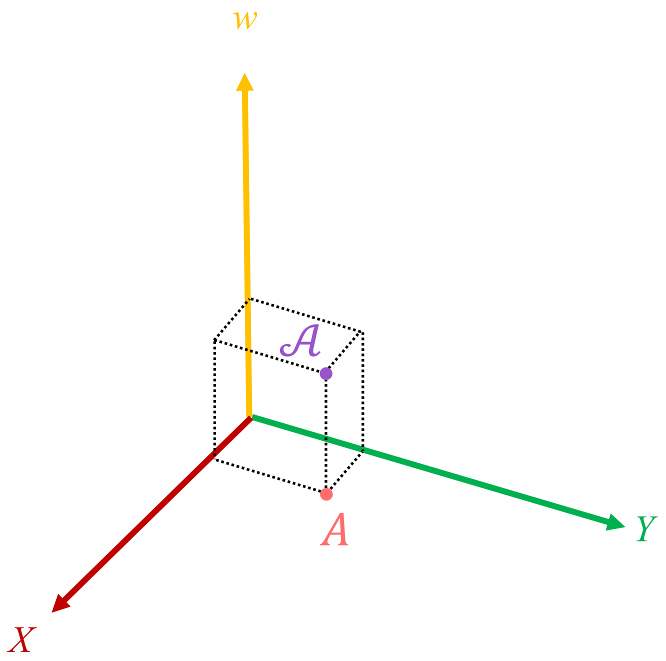
## Geometric interpretation

- Let $A = (x, y)$ a vector within a 2D Euclidean space



# Geometric interpretation

- Let $A = (x, y)$ a vector within a 2D Euclidean space
- $\mathcal{A} = [x \quad y \quad 1]^\top$ represents $A$ within the homogeneous space



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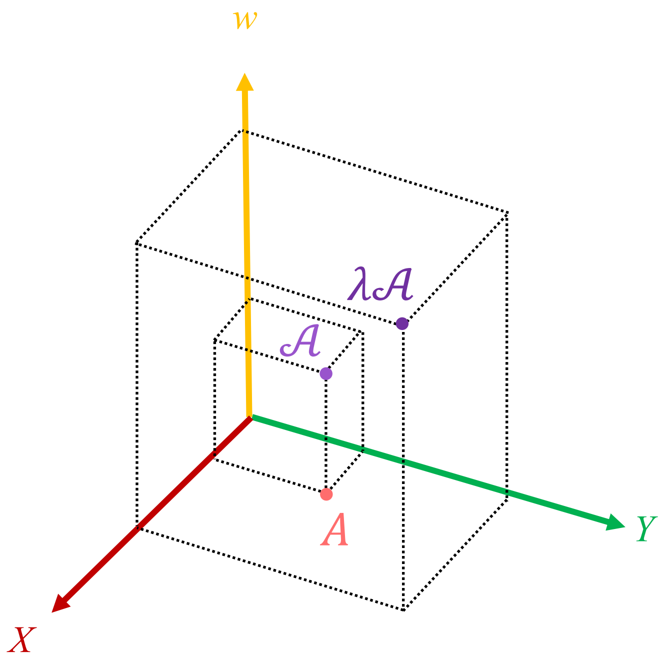
- Let $A = (x, y)$ a vector within a 2D Euclidean space
- $\mathcal{A} = [x \quad y \quad 1]^\mathsf{T}$ represents $A$ within the homogeneous space
- $\lambda\mathcal{A} = [\lambda x \quad \lambda y \quad \lambda]^\mathsf{T}$ is another representation of $A$



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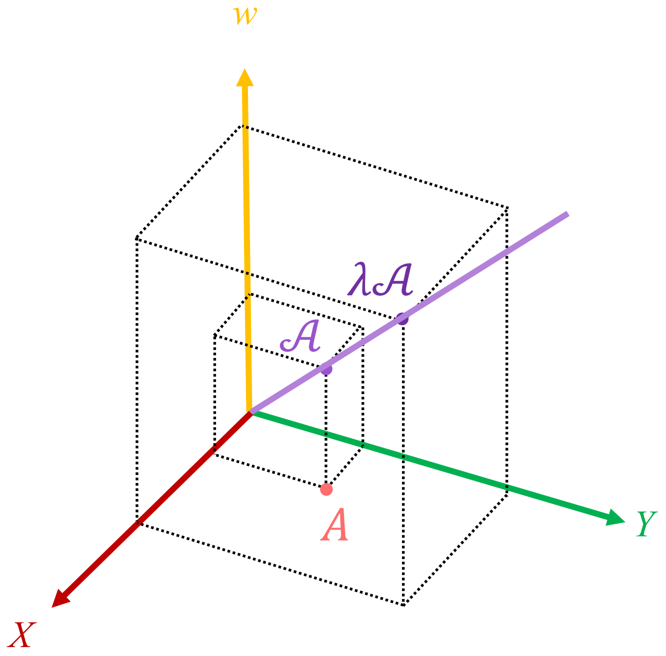


All homogeneous representations of $A$ are lying on the same line

## Homogeneous transformation

- Let be a homogeneous space of dimension $n$ and let $\mathcal{V}$ be a homogeneous vector. A **homogeneous transform** is an application defined such as:

$$\mathcal{H}(\mathcal{V}) = \mathrm{HV}$$

With:

- $\mathrm{H}$ is a $n \times n$ square matrix that represents $\mathcal{H}$
- $V$ is a $n$ dimensioned vector that represents $\mathcal{V}$
- $\mathrm{HV}$ is the matrix product of $\mathrm{H}$ and $\mathrm{V}$

# Homogeneous transformation

- Within 4 dimensioned homogeneous space
  - $\mathcal{V} = [x \quad y \quad z \quad w]^{\mathrm{T}}$
  - H is such as:

$$\mathrm{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix}$$

# Homogeneous transformation

- Identity transform

$$\mathrm{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Computation

$$\mathrm{H}\mathcal{V} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

# Homogeneous transformation

- Homogeneous transformation linearity
  - Let $\mathcal{H}$ be a homogeneous transform represented by matrix $\mathrm{H}$:

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix}$$

  - Let $\mathcal{A}$ and $\mathcal{B}$ two homogeneous vectors represented by column matrices $\mathrm{A}$ and $\mathrm{B}$ respectively and let $\lambda$ and $\mu$ be two scalars:

$$\mathrm{A} = \begin{bmatrix} x_a \\ y_a \\ z_a \\ w_a \end{bmatrix} \quad \mathrm{B} = \begin{bmatrix} x_b \\ y_b \\ z_b \\ w_b \end{bmatrix} \quad \lambda\mathrm{A} = \begin{bmatrix} \lambda x_a \\ \lambda y_a \\ \lambda z_a \\ \lambda w_a \end{bmatrix} \quad \mu\mathrm{B} = \begin{bmatrix} \mu x_b \\ \mu y_b \\ \mu z_b \\ \mu w_b \end{bmatrix}$$

# Homogeneous transformation

- Homogeneous transformation linearity

$$\mathcal{H}(\lambda\mathcal{A} + \mu\mathcal{B}) = \mathrm{H}(\lambda\mathrm{A} + \mu\mathrm{B})$$

# Homogeneous transformation

- Homogeneous transformation linearity

$$\mathcal{H}(\lambda\mathcal{A} + \mu\mathcal{B}) = \mathrm{H}(\lambda\mathrm{A} + \mu\mathrm{B})$$

$$\mathcal{H}(\lambda\mathcal{A} + \mu\mathcal{B}) = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix} \begin{bmatrix} \lambda x_a + \mu x_b \\ \lambda y_a + \mu y_b \\ \lambda z_a + \mu z_b \\ \lambda w_a + \mu w_b \end{bmatrix}$$

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- Homogeneous transformation linearity

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$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix} \begin{bmatrix} \lambda x_a + \mu x_b \\ \lambda y_a + \mu y_b \\ \lambda z_a + \mu z_b \\ \lambda w_a + \mu w_b \end{bmatrix}$$

Matrix product

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \begin{bmatrix} h_{11}(\lambda x_a + \mu x_b) + h_{12}(\lambda y_a + \mu y_b) + h_{13}(\lambda z_a + \mu z_b) + h_{14}(\lambda w_a + \mu w_b) \\ h_{21}(\lambda x_a + \mu x_b) + h_{22}(\lambda y_a + \mu y_b) + h_{23}(\lambda z_a + \mu z_b) + h_{24}(\lambda w_a + \mu w_b) \\ h_{31}(\lambda x_a + \mu x_b) + h_{32}(\lambda y_a + \mu y_b) + h_{33}(\lambda z_a + \mu z_b) + h_{44}(\lambda w_a + \mu w_b) \\ h_{41}(\lambda x_a + \mu x_b) + h_{42}(\lambda y_a + \mu y_b) + h_{43}(\lambda z_a + \mu z_b) + h_{44}(\lambda w_a + \mu w_b) \end{bmatrix}$$

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- Homogeneous transformation linearity

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix} \begin{bmatrix} \lambda x_a + \mu x_b \\ \lambda y_a + \mu y_b \\ \lambda z_a + \mu z_b \\ \lambda w_a + \mu w_b \end{bmatrix}$$

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \begin{bmatrix} h_{11}(\lambda x_a + \mu x_b) + h_{12}(\lambda y_a + \mu y_b) + h_{13}(\lambda z_a + \mu z_b) + h_{14}(\lambda w_a + \mu w_b) \\ h_{21}(\lambda x_a + \mu x_b) + h_{22}(\lambda y_a + \mu y_b) + h_{23}(\lambda z_a + \mu z_b) + h_{24}(\lambda w_a + \mu w_b) \\ h_{31}(\lambda x_a + \mu x_b) + h_{32}(\lambda y_a + \mu y_b) + h_{33}(\lambda z_a + \mu z_b) + h_{44}(\lambda w_a + \mu w_b) \\ h_{41}(\lambda x_a + \mu x_b) + h_{42}(\lambda y_a + \mu y_b) + h_{43}(\lambda z_a + \mu z_b) + h_{44}(\lambda w_a + \mu w_b) \end{bmatrix}$$

Distribution

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \begin{bmatrix} h_{11}\lambda x_a + h_{11}\mu x_b + h_{12}\lambda y_a + h_{12}\mu y_b + h_{13}\lambda z_a + h_{13}\mu z_b + h_{14}\lambda w_a + h_{14}\mu w_b \\ h_{21}\lambda x_a + h_{21}\mu x_b + h_{22}\lambda y_a + h_{22}\mu y_b + h_{23}\lambda z_a + h_{23}\mu z_b + h_{24}\lambda w_a + h_{24}\mu w_b \\ h_{31}\lambda x_a + h_{31}\mu x_b + h_{32}\lambda y_a + h_{32}\mu y_b + h_{33}\lambda z_a + h_{33}\mu z_b + h_{44}\lambda w_a + h_{34}\mu w_b \\ h_{41}\lambda x_a + h_{31}\mu x_b + h_{42}\lambda y_a + h_{42}\mu y_b + h_{43}\lambda z_a + h_{43}\mu z_b + h_{44}\lambda w_a + h_{44}\mu w_b \end{bmatrix}$$

# Homogeneous transformation

- Homogeneous transformation linearity

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \begin{bmatrix} h_{11}(\lambda x_a+\mu x_b)+h_{12}(\lambda y_a+\mu y_b)+h_{13}(\lambda z_a+\mu z_b)+h_{14}(\lambda w_a+\mu w_b) \\ h_{21}(\lambda x_a+\mu x_b)+h_{22}(\lambda y_a+\mu y_b)+h_{23}(\lambda z_a+\mu z_b)+h_{24}(\lambda w_a+\mu w_b) \\ h_{31}(\lambda x_a+\mu x_b)+h_{32}(\lambda y_a+\mu y_b)+h_{33}(\lambda z_a+\mu z_b)+h_{44}(\lambda w_a+\mu w_b) \\ h_{41}(\lambda x_a+\mu x_b)+h_{42}(\lambda y_a+\mu y_b)+h_{43}(\lambda z_a+\mu z_b)+h_{44}(\lambda w_a+\mu w_b) \end{bmatrix}$$

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \begin{bmatrix} h_{11}\lambda x_a+h_{11}\mu x_b+h_{12}\lambda y_a+h_{12}\mu y_b+h_{13}\lambda z_a+h_{13}\mu z_b+h_{14}\lambda w_a+h_{14}\mu w_b \\ h_{21}\lambda x_a+h_{21}\mu x_b+h_{22}\lambda y_a+h_{22}\mu y_b+h_{23}\lambda z_a+h_{23}\mu z_b+h_{24}\lambda w_a+h_{24}\mu w_b \\ h_{31}\lambda x_a+h_{31}\mu x_b+h_{32}\lambda y_a+h_{32}\mu y_b+h_{33}\lambda z_a+h_{33}\mu z_b+h_{44}\lambda w_a+h_{34}\mu w_b \\ h_{41}\lambda x_a+h_{31}\mu x_b+h_{42}\lambda y_a+h_{42}\mu y_b+h_{43}\lambda z_a+h_{43}\mu z_b+h_{44}\lambda w_a+h_{44}\mu w_b \end{bmatrix}$$

Grouping

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \begin{bmatrix} h_{11}\lambda x_a+h_{12}\lambda y_a+h_{13}\lambda z_a+h_{14}\lambda w_a+h_{11}\mu x_b+h_{12}\mu y_b+h_{13}\mu z_b+h_{14}\mu w_b \\ h_{21}\lambda x_a+h_{22}\lambda y_a+h_{23}\lambda z_a+h_{24}\lambda w_a+h_{21}\mu x_b+h_{22}\mu y_b+h_{23}\mu z_b+h_{24}\mu w_b \\ h_{31}\lambda x_a+h_{32}\lambda y_a+h_{33}\lambda z_a+h_{34}\lambda w_a+h_{31}\mu x_b+h_{32}\mu y_b+h_{33}\mu z_b+h_{34}\mu w_b \\ h_{41}\lambda x_a+h_{42}\lambda y_a+h_{43}\lambda z_a+h_{44}\lambda w_a+h_{41}\mu x_b+h_{42}\mu y_b+h_{43}\mu z_b+h_{44}\mu w_b \end{bmatrix}$$

# Homogeneous transformation

- Homogeneous transformation linearity

$$\mathcal{H}(\lambda\mathcal{A} + \mu\mathcal{B}) = \begin{bmatrix} h_{11}\lambda x_a + h_{11}\mu x_b + h_{12}\lambda y_a + h_{12}\mu y_b + h_{13}\lambda z_a + h_{13}\mu z_b + h_{14}\lambda w_a + h_{14}\mu w_b \\ h_{21}\lambda x_a + h_{21}\mu x_b + h_{22}\lambda y_a + h_{22}\mu y_b + h_{23}\lambda z_a + h_{23}\mu z_b + h_{24}\lambda w_a + h_{24}\mu w_b \\ h_{31}\lambda x_a + h_{31}\mu x_b + h_{32}\lambda y_a + h_{32}\mu y_b + h_{33}\lambda z_a + h_{33}\mu z_b + h_{44}\lambda w_a + h_{34}\mu w_b \\ h_{41}\lambda x_a + h_{31}\mu x_b + h_{42}\lambda y_a + h_{42}\mu y_b + h_{43}\lambda z_a + h_{43}\mu z_b + h_{44}\lambda w_a + h_{44}\mu w_b \end{bmatrix}$$

$$\mathcal{H}(\lambda\mathcal{A} + \mu\mathcal{B}) = \begin{bmatrix} h_{11}\lambda x_a + h_{12}\lambda y_a + h_{13}\lambda z_a + h_{14}\lambda w_a + h_{11}\mu x_b + h_{12}\mu y_b + h_{13}\mu z_b + h_{14}\mu w_b \\ h_{21}\lambda x_a + h_{22}\lambda y_a + h_{23}\lambda z_a + h_{24}\lambda w_a + h_{21}\mu x_b + h_{22}\mu y_b + h_{23}\mu z_b + h_{24}\mu w_b \\ h_{31}\lambda x_a + h_{32}\lambda y_a + h_{33}\lambda z_a + h_{34}\lambda w_a + h_{31}\mu x_b + h_{32}\mu y_b + h_{33}\mu z_b + h_{34}\mu w_b \\ h_{41}\lambda x_a + h_{42}\lambda y_a + h_{43}\lambda z_a + h_{44}\lambda w_a + h_{41}\mu x_b + h_{42}\mu y_b + h_{43}\mu z_b + h_{44}\mu w_b \end{bmatrix}$$

Decomposition

$$\mathcal{H}(\lambda\mathcal{A} + \mu\mathcal{B}) = \begin{bmatrix} h_{11}\lambda x_a + h_{12}\lambda y_a + h_{13}\lambda z_a + h_{14}\lambda w_a \\ h_{21}\lambda x_a + h_{22}\lambda y_a + h_{23}\lambda z_a + h_{24}\lambda w_a \\ h_{31}\lambda x_a + h_{32}\lambda y_a + h_{33}\lambda z_a + h_{34}\lambda w_a \\ h_{41}\lambda x_a + h_{42}\lambda y_a + h_{43}\lambda z_a + h_{44}\lambda w_a \end{bmatrix} + \begin{bmatrix} h_{11}\mu x_b + h_{12}\mu y_b + h_{13}\mu z_b + h_{14}\mu w_b \\ h_{21}\mu x_b + h_{22}\mu y_b + h_{23}\mu z_b + h_{24}\mu w_b \\ h_{31}\mu x_b + h_{32}\mu y_b + h_{33}\mu z_b + h_{34}\mu w_b \\ h_{41}\mu x_b + h_{42}\mu y_b + h_{43}\mu z_b + h_{44}\mu w_b \end{bmatrix}$$

# Homogeneous transformation

- Homogeneous transformation linearity

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B})=\begin{bmatrix} h_{11}\lambda x_a+h_{12}\lambda y_a+h_{13}\lambda z_a+h_{14}\lambda w_a+h_{11}\mu x_b+h_{12}\mu y_b+h_{13}\mu z_b+h_{14}\mu w_b \\ h_{21}\lambda x_a+h_{22}\lambda y_a+h_{23}\lambda z_a+h_{24}\lambda w_a+h_{21}\mu x_b+h_{22}\mu y_b+h_{23}\mu z_b+h_{24}\mu w_b \\ h_{31}\lambda x_a+h_{32}\lambda y_a+h_{33}\lambda z_a+h_{34}\lambda w_a+h_{31}\mu x_b+h_{32}\mu y_b+h_{33}\mu z_b+h_{34}\mu w_b \\ h_{41}\lambda x_a+h_{42}\lambda y_a+h_{43}\lambda z_a+h_{44}\lambda w_a+h_{41}\mu x_b+h_{42}\mu y_b+h_{43}\mu z_b+h_{44}\mu w_b \end{bmatrix}$$

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B})=\begin{bmatrix} h_{11}\lambda x_a+h_{12}\lambda y_a+h_{13}\lambda z_a+h_{14}\lambda w_a \\ h_{21}\lambda x_a+h_{22}\lambda y_a+h_{23}\lambda z_a+h_{24}\lambda w_a \\ h_{31}\lambda x_a+h_{32}\lambda y_a+h_{33}\lambda z_a+h_{34}\lambda w_a \\ h_{41}\lambda x_a+h_{42}\lambda y_a+h_{43}\lambda z_a+h_{44}\lambda w_a \end{bmatrix}+\begin{bmatrix} h_{11}\mu x_b+h_{12}\mu y_b+h_{13}\mu z_b+h_{14}\mu w_b \\ h_{21}\mu x_b+h_{22}\mu y_b+h_{23}\mu z_b+h_{24}\mu w_b \\ h_{31}\mu x_b+h_{32}\mu y_b+h_{33}\mu z_b+h_{34}\mu w_b \\ h_{41}\mu x_b+h_{42}\mu y_b+h_{43}\mu z_b+h_{44}\mu w_b \end{bmatrix}$$

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B})=\lambda\begin{bmatrix} h_{11}x_a+h_{12}y_a+h_{13}z_a+h_{14}w_a \\ h_{21}x_a+h_{22}y_a+h_{23}z_a+h_{24}w_a \\ h_{31}x_a+h_{32}y_a+h_{33}z_a+h_{34}w_a \\ h_{41}x_a+h_{42}y_a+h_{43}z_a+h_{44}w_a \end{bmatrix}+\mu\begin{bmatrix} h_{11}x_b+h_{12}y_b+h_{13}z_b+h_{14}w_b \\ h_{21}x_b+h_{22}y_b+h_{23}z_b+h_{24}w_b \\ h_{31}x_b+h_{32}y_b+h_{33}z_b+h_{34}w_b \\ h_{41}x_b+h_{42}y_b+h_{43}z_b+h_{44}w_b \end{bmatrix}$$

Factorization

# Homogeneous transformation

- Homogeneous transformation linearity

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \begin{bmatrix} h_{11}\lambda x_a + h_{12}\lambda y_a + h_{13}\lambda z_a + h_{14}\lambda w_a \\ h_{21}\lambda x_a + h_{22}\lambda y_a + h_{23}\lambda z_a + h_{24}\lambda w_a \\ h_{31}\lambda x_a + h_{32}\lambda y_a + h_{33}\lambda z_a + h_{34}\lambda w_a \\ h_{41}\lambda x_a + h_{42}\lambda y_a + h_{43}\lambda z_a + h_{44}\lambda w_a \end{bmatrix} + \begin{bmatrix} h_{11}\mu x_b + h_{12}\mu y_b + h_{13}\mu z_b + h_{14}\mu w_b \\ h_{21}\mu x_b + h_{22}\mu y_b + h_{23}\mu z_b + h_{24}\mu w_b \\ h_{31}\mu x_b + h_{32}\mu y_b + h_{33}\mu z_b + h_{34}\mu w_b \\ h_{41}\mu x_b + h_{42}\mu y_b + h_{43}\mu z_b + h_{44}\mu w_b \end{bmatrix}$$

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \lambda\begin{bmatrix} h_{11}x_a + h_{12}y_a + h_{13}z_a + h_{14}w_a \\ h_{21}x_a + h_{22}y_a + h_{23}z_a + h_{24}w_a \\ h_{31}x_a + h_{32}y_a + h_{33}z_a + h_{34}w_a \\ h_{41}x_a + h_{42}y_a + h_{43}z_a + h_{44}w_a \end{bmatrix} + \mu\begin{bmatrix} h_{11}x_b + h_{12}y_b + h_{13}z_b + h_{14}w_b \\ h_{21}x_b + h_{22}y_b + h_{23}z_b + h_{24}w_b \\ h_{31}x_b + h_{32}y_b + h_{33}z_b + h_{34}w_b \\ h_{41}x_b + h_{42}y_b + h_{43}z_b + h_{44}w_b \end{bmatrix}$$

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \lambda \mathrm{HA} + \mu \mathrm{HB}$$

Matrix product

# Homogeneous transformation

- Homogeneous transformation linearity

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \lambda\begin{bmatrix} h_{11}x_a + h_{12}y_a + h_{13}z_a + h_{14}w_a \\ h_{21}x_a + h_{22}y_a + h_{23}z_a + h_{24}w_a \\ h_{31}x_a + h_{32}y_a + h_{33}z_a + h_{34}w_a \\ h_{41}x_a + h_{42}y_a + h_{43}z_a + h_{44}w_a \end{bmatrix} + \mu\begin{bmatrix} h_{11}x_b + h_{12}y_b + h_{13}z_b + h_{14}w_b \\ h_{21}x_b + h_{22}y_b + h_{23}z_b + h_{24}w_b \\ h_{31}x_b + h_{32}y_b + h_{33}z_b + h_{34}w_b \\ h_{41}x_b + h_{42}y_b + h_{43}z_b + h_{44}w_b \end{bmatrix}$$

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \lambda \mathrm{HA} + \mu \mathrm{HB}$$

Definition

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \lambda\mathcal{H}(\mathcal{A}) + \mu\mathcal{H}(\mathcal{B})$$

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- Homogeneous transformation linearity

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \lambda\begin{bmatrix} h_{11}x_a + h_{12}y_a + h_{13}z_a + h_{14}w_a \\ h_{21}x_a + h_{22}y_a + h_{23}z_a + h_{24}w_a \\ h_{31}x_a + h_{32}y_a + h_{33}z_a + h_{34}w_a \\ h_{41}x_a + h_{42}y_a + h_{43}z_a + h_{44}w_a \end{bmatrix} + \mu\begin{bmatrix} h_{11}x_b + h_{12}y_b + h_{13}z_b + h_{14}w_b \\ h_{21}x_b + h_{22}y_b + h_{23}z_b + h_{24}w_b \\ h_{31}x_b + h_{32}y_b + h_{33}z_b + h_{34}w_b \\ h_{41}x_b + h_{42}y_b + h_{43}z_b + h_{44}w_b \end{bmatrix}$$

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \lambda\mathrm{H}\mathcal{A} + \mu\mathrm{H}\mathcal{B}$$

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \lambda\mathcal{H}(\mathcal{A}) + \mu\mathcal{H}(\mathcal{B})$$

Any homogeneous transformation is a **linear application**

# Homogeneous Transformation

- Which kind of transformation can be represented ?
- Is it possible to use Homogeneous transformations to represents Euclidean transformations ?
- Is there transformation that can only be computed using Homogeneous coordinates ?

# Homogeneous Translation

- Let $\mathcal{X} = [x, y, z, w]^\top$ be a homogeneous vector. The translation of $\mathcal{X}$ along a vector $[\alpha, \beta, \gamma, 1]^\top$, denoted $\mathcal{T}(\alpha, \beta, \gamma)$, is such as:

$$\mathcal{T}(\alpha, \beta, \gamma)(\mathcal{X}) = \mathrm{T}\mathcal{X}$$

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$$\mathcal{T}(\alpha, \beta, \gamma)(\mathcal{X}) = \mathrm{T}\mathcal{X}$$

Where:

$$\mathrm{T} = \begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\mathcal{T}(\alpha, \beta, \gamma)(\mathcal{X}) = \mathrm{T}\mathcal{X} = \begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x + \alpha \\ y + \beta \\ z + \gamma \\ w \end{bmatrix}$$

## Computing Euclidean translation

- Let $X = (x, y, z)$ be a vector within 3D Euclidean space.

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- The homogeneous translation of $\mathcal{X}$ along $(\alpha, \beta, \gamma)$:

$$\mathcal{T}(\alpha,\beta,\gamma)(\mathcal{X}) = \begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x+\alpha \\ y+\beta \\ z+\gamma \\ 1 \end{bmatrix}$$

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- Euclidean representation: $[x+\alpha, y+\beta, z+\gamma, 1]^T \to \left(\frac{x+\alpha}{1}, \frac{y+\beta}{1}, \frac{z+\gamma}{1}\right) = (x+\alpha, y+\beta, z+\gamma)$

## Computing Euclidean translation

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$$\boldsymbol{\mathcal{T}}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})(\boldsymbol{\mathcal{X}}) \rightarrow \boldsymbol{T}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})(\boldsymbol{X})$$

## Homogeneous translation inverse

- Let $\mathcal{X}$ and $\mathcal{Y}$ be a homogeneous vectors where $\mathcal{Y}$ is the result of the homogeneous translation of $\mathcal{X}$ along a vector $[\alpha, \beta, \gamma, 1]^\top$

$$\mathcal{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \qquad \mathcal{Y} = \mathcal{T}(\alpha, \beta, \gamma)(\mathcal{X}) = \mathrm{T}\mathcal{X} = \begin{bmatrix} x + \alpha w \\ y + \beta w \\ z + \gamma w \\ w \end{bmatrix}$$

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$$\mathcal{T}(-\alpha, -\beta, -\gamma)(\mathcal{Y}) = T'\mathcal{Y} = \begin{bmatrix} 1 & 0 & 0 & -\alpha \\ 0 & 1 & 0 & -\beta \\ 0 & 0 & 1 & -\gamma \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x + \alpha \\ y + \beta \\ z + \gamma \\ w \end{bmatrix}$$

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# Homogeneous translation inverse

- Let $\mathcal{X}$ and $\mathcal{Y}$ be a homogeneous vectors where $\mathcal{Y}$ is the result of the homogeneous translation of $\mathcal{X}$ along a vector $[\alpha, \beta, \gamma, 1]^\top$

$$\mathcal{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \qquad \mathcal{Y} = \mathcal{T}(\alpha, \beta, \gamma)(\mathcal{X}) = \mathrm{T}\mathcal{X} = \begin{bmatrix} x + \alpha w \\ y + \beta w \\ z + \gamma w \\ w \end{bmatrix}$$

$$\mathcal{T}(-\alpha, -\beta, -\gamma)(\mathcal{Y}) = T'\mathcal{Y} = \begin{bmatrix} 1 & 0 & 0 & -\alpha \\ 0 & 1 & 0 & -\beta \\ 0 & 0 & 1 & -\gamma \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x + \alpha \\ y + \beta \\ z + \gamma \\ w \end{bmatrix} = \begin{bmatrix} x + \cancel{\alpha w} - \cancel{\alpha w} \\ y + \cancel{\beta w} - \cancel{\beta w} \\ z + \cancel{\gamma w} - \cancel{\gamma w} \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

# Homogeneous translation inverse

- Let $\mathcal{X}$ and $\mathcal{Y}$ be a homogeneous vectors where $\mathcal{Y}$ is the result of the homogeneous translation of $\mathcal{X}$ along a vector $[\alpha, \beta, \gamma, 1]^{\top}$

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$$\mathcal{T}(-\alpha, -\beta, -\gamma)\big(\mathcal{T}(\alpha, \beta, \gamma)(\mathcal{X})\big) = T'T\mathcal{X} = \mathcal{X}$$

A Homogeneous translation $\mathcal{T}(\alpha, \beta, \gamma)$ is invertible and $\mathcal{T}(-\alpha, -\beta, -\gamma)$ is its inverse

# Homogeneous translation inverse

- Matrix inverse method

$$T = \begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T' = \begin{bmatrix} 1 & 0 & 0 & -\alpha \\ 0 & 1 & 0 & -\beta \\ 0 & 0 & 1 & -\gamma \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\mathrm{TT}' = ID$$

$\mathrm{T}'$ is the inverse matrix of $\mathrm{T}$

# Homogeneous translation matrix structure

$$\mathrm{H} = \begin{bmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

# Homogeneous translation

- Definition

$$\mathcal{T}(\alpha,\beta,\gamma) = \mathrm{T} = \begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Properties
  - Linear application
  - Invertible: $\mathcal{T}^{-1}(\alpha,\beta,\gamma) = T^{-1} = \begin{bmatrix} 1 & 0 & 0 & -\alpha \\ 0 & 1 & 0 & -\beta \\ 0 & 0 & 1 & -\gamma \\ 0 & 0 & 0 & 1 \end{bmatrix}$

## Homogeneous axis rotation

- Let $\mathcal{V} = [x, y, z, w]^T$ be a homogeneous vector. The rotation of $\mathcal{V}$ by an angle $\omega$ around $X$ axis, denoted $\mathcal{R}_x(\omega)$, is such as:

$$\mathcal{R}_x(\omega)(\mathcal{V}) = \mathrm{R}_x\mathcal{V}$$

Where:

$$\mathrm{R}_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ 0 & \sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\mathcal{R}_x(\omega)(\mathcal{V}) = \begin{bmatrix} x \\ y\cos\omega - z\sin\omega \\ y\sin\omega + z\cos\omega \\ w \end{bmatrix}$$

# Computing Euclidean axis rotation

- Let $V = (x, y, z)$ be a vector within 3D Euclidean space.

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$$\rightarrow (x, y\cos\omega - z\sin\omega\,, y\sin\omega + z\cos\omega)$$

$$\mathcal{R}_x(\omega)(\mathcal{V}) \rightarrow R_x(\omega)(V)$$

## Homogeneous rotation inverse

- Let $\mathrm{R}_x$ be the matrix of the homogeneous rotation $\mathcal{R}_x(\omega)$ and let $\mathrm{R}'_x$ be the matrix of the homogeneous rotation $\mathcal{R}_x(-\omega)$

$$\mathrm{R}_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ 0 & \sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{R}'_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-\omega) & -\sin(-\omega) & 0 \\ 0 & \sin(-\omega) & \cos(-\omega) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\cos(-\omega) = \cos\omega$$

$$\mathrm{R}'_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & -\sin(-\omega) & 0 \\ 0 & \sin(-\omega) & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\sin(-\omega) = -\sin\omega$$

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# Homogeneous rotation inverse

- Let $\mathrm{R}_x$ be the matrix of the homogeneous rotation $\mathcal{R}_x(\omega)$ and let $\mathrm{R}_x^\top$ be the matrix of the homogeneous rotation $\mathcal{R}_x(-\omega)$

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$$\mathrm{R}_x\mathrm{R}_x^\top = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ 0 & \sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\mathrm{R}_x\mathrm{R}_x^\top = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2\omega + \sin^2\omega & \cos\omega\sin\omega - \cos\omega\sin\omega & 0 \\ 0 & \sin\omega\cos\omega - \cos\omega\sin\omega & \sin^2\omega + \cos^2\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\sin^2\omega + \cos^2\omega = 1$$

$$\mathrm{R}_x\mathrm{R}_x^\top = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \cos\omega\sin\omega - \cos\omega\sin\omega & 0 \\ 0 & \sin\omega\cos\omega - \cos\omega\sin\omega & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## Homogeneous rotation inverse

- Let $\mathrm{R}_x$ be the matrix of the homogeneous rotation $\mathcal{R}_x(\omega)$ and let $\mathrm{R}_x^\top$ be the matrix of the homogeneous rotation $\mathcal{R}_x(-\omega)$

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A Homogeneous $X$ axis rotation $\mathcal{R}_x(\omega)$ is invertible and $\mathcal{R}_x(-\omega)$ is its inverse. Moreover, $\mathrm{R}_x^{-1} = \mathrm{R}_x^T$

# Homogeneous axis rotation

- Let $\mathcal{V} = [x, y, z, w]^\top$ be a homogeneous vector. The rotation of $\mathcal{V}$ by an angle φ around $Y$ axis, denoted $\mathcal{R}_y(\varphi)$, is such as:

$$\mathcal{R}_y(\varphi)(\mathcal{V}) = \mathrm{R}_y\mathcal{V}$$

Where:

$$\mathrm{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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- Let $V = (x, y, z)$ be a vector within 3D Euclidean space.

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- Homogeneous representation of $V$ is: $\mathcal{V} = [x, y, z, 1]^\top$
- Homogeneous rotation of $\mathcal{V}$ around $X$ axis by an angle $\omega$:

$$\mathcal{R}_y(\varphi)(\mathcal{V}) = \begin{bmatrix} x\cos\varphi + z\sin\varphi \\ y \\ z\cos\varphi - x\sin\varphi \\ 1 \end{bmatrix} \rightarrow \left(\frac{x\cos\varphi + z\sin\varphi}{1}, \frac{y}{1}, \frac{z\cos\varphi - x\sin\varphi}{1}\right)$$

$$\rightarrow (x\cos\varphi + z\sin\varphi\,, y, z\cos\varphi - x\sin\varphi)$$

## Computing Euclidean axis rotation

- Let $V = (x, y, z)$ be a vector within 3D Euclidean space.
- Homogeneous representation of $V$ is: $\mathcal{V} = [x, y, z, 1]^\mathsf{T}$
- Homogeneous rotation of $\mathcal{V}$ around $X$ axis by an angle $\omega$:

$$\mathcal{R}_y(\varphi)(\mathcal{V}) = \begin{bmatrix} x\cos\varphi + z\sin\varphi \\ y \\ z\cos\varphi - x\sin\varphi \\ 1 \end{bmatrix} \rightarrow \left(\frac{x\cos\varphi + z\sin\varphi}{1}, \frac{y}{1}, \frac{z\cos\varphi - x\sin\varphi}{1}\right)$$

$$\rightarrow (x\cos\varphi + z\sin\varphi, y, z\cos\varphi - x\sin\varphi)$$

$$\mathcal{R}_y(\varphi)(\mathcal{V}) \rightarrow R_y(\varphi)(V)$$

# Homogeneous rotation inverse

- Let $\mathrm{R}_y$ be the matrix of the homogeneous rotation $\mathcal{R}_y(\varphi)$ and let $\mathrm{R}'_y$ be the matrix of the homogeneous rotation $\mathcal{R}_y(-\varphi)$

$$\mathrm{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{R}'_y = \begin{bmatrix} \cos(-\varphi) & 0 & \sin(-\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(-\varphi) & 0 & \cos(-\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Homogeneous rotation inverse

- Let $\mathrm{R}_y$ be the matrix of the homogeneous rotation $\mathcal{R}_y(\varphi)$ and let $\mathrm{R}'_y$ be the matrix of the homogeneous rotation $\mathcal{R}_y(-\varphi)$

$$\mathrm{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{R}'_y = \begin{bmatrix} \cos(-\varphi) & 0 & \sin(-\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(-\varphi) & 0 & \cos(-\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cos(-\omega) = \cos\omega$$

$$\mathrm{R}'_y = \begin{bmatrix} \cos\varphi & 0 & \sin(-\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(-\varphi) & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Homogeneous rotation inverse

- Let $\mathrm{R}_y$ be the matrix of the homogeneous rotation $\mathcal{R}_y(\varphi)$ and let $\mathrm{R}'_y$ be the matrix of the homogeneous rotation $\mathcal{R}_y(-\varphi)$

$$\mathrm{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{R}'_y = \begin{bmatrix} \cos\varphi & 0 & \sin(-\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(-\varphi) & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sin(-\omega) = -\sin\omega$$

$$\mathrm{R}'_y = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## Homogeneous rotation inverse

- Let $\mathrm{R}_y$ be the matrix of the homogeneous rotation $\mathcal{R}_y(\varphi)$ and let $\mathrm{R}'_y$ be the matrix of the homogeneous rotation $\mathcal{R}_y(-\varphi)$

$$\mathrm{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{R}'_y = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## Homogeneous rotation inverse

- Let $\mathrm{R}_y$ be the matrix of the homogeneous rotation $\mathcal{R}_y(\varphi)$ and let $\mathrm{R}'_y$ be the matrix of the homogeneous rotation $\mathcal{R}_y(-\varphi)$

$$\mathrm{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{R}'_y = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{R}_y^\top$$

# Homogeneous rotation inverse

- Let $\mathrm{R}_y$ be the matrix of the homogeneous rotation $\mathcal{R}_y(\varphi)$ and let $\mathrm{R}_y^\top$ be the matrix of the homogeneous rotation $\mathcal{R}_y(-\varphi)$

$$\mathrm{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{R}_y^\top = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## Homogeneous rotation inverse

- Let $\mathrm{R}_y$ be the matrix of the homogeneous rotation $\mathcal{R}_y(\varphi)$ and let $\mathrm{R}_y^\top$ be the matrix of the homogeneous rotation $\mathcal{R}_y(-\varphi)$

$$\mathrm{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{R}_y^\top = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{R}_y\mathrm{R}_y^\top = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Homogeneous rotation inverse

- Let $\mathrm{R}_y$ be the matrix of the homogeneous rotation $\mathcal{R}_y(\varphi)$ and let $\mathrm{R}_y^\top$ be the matrix of the homogeneous rotation $\mathcal{R}_y(-\varphi)$

$$\mathrm{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{R}_y^\top = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{R}_y\mathrm{R}_y^\top = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{R}_y\mathrm{R}_y^\top = \begin{bmatrix} \cos^2\varphi + \sin^2\varphi & 0 & -\cos\varphi\sin\varphi + \sin\varphi\cos\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi\cos\varphi + \cos\varphi\sin\varphi & 0 & \cos^2\varphi + \sin^2\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## Homogeneous rotation inverse

- Let $\mathrm{R}_y$ be the matrix of the homogeneous rotation $\mathcal{R}_y(\varphi)$ and let $\mathrm{R}_y^\top$ be the matrix of the homogeneous rotation $\mathcal{R}_y(-\varphi)$

$$\mathrm{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{R}_y^\top = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{R}_y\mathrm{R}_y^\top = \begin{bmatrix} \cos^2\varphi + \sin^2\varphi & 0 & -\cos\varphi\sin\varphi + \sin\varphi\cos\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi\cos\varphi + \cos\varphi\sin\varphi & 0 & \cos^2\varphi + \sin^2\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sin^2\omega + \cos^2\omega = 1$$

$$\mathrm{R}_y\mathrm{R}_y^\top = \begin{bmatrix} 1 & 0 & -\cos\varphi\sin\varphi + \sin\varphi\cos\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi\cos\varphi + \cos\varphi\sin\varphi & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## Homogeneous rotation inverse

- Let $\mathrm{R}_y$ be the matrix of the homogeneous rotation $\mathcal{R}_y(\varphi)$ and let $\mathrm{R}_y^\top$ be the matrix of the homogeneous rotation $\mathcal{R}_y(-\varphi)$

$$\mathrm{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{R}_y^\top = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{R}_y\mathrm{R}_y^\top = \begin{bmatrix} 1 & 0 & -\cos\varphi\sin\varphi + \sin\varphi\cos\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi\cos\varphi + \cos\varphi\sin\varphi & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{R}_y\mathrm{R}_y^\top = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Homogeneous rotation inverse

- Let $\mathrm{R}_y$ be the matrix of the homogeneous rotation $\mathcal{R}_y(\varphi)$ and let $\mathrm{R}_y^\top$ be the matrix of the homogeneous rotation $\mathcal{R}_y(-\varphi)$

$$\mathrm{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{R}_y^\top = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{R}_y\mathrm{R}_y^\top = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = ID$$

# Homogeneous rotation inverse

- Let $\mathrm{R}_y$ be the matrix of the homogeneous rotation $\mathcal{R}_y(\varphi)$ and let $\mathrm{R}_y^\top$ be the matrix of the homogeneous rotation $\mathcal{R}_y(-\varphi)$

$$\mathrm{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{R}_y^\top = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{R}_y\mathrm{R}_y^\top = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = ID$$

A Homogeneous $Y$ axis rotation $\mathcal{R}_y(\varphi)$ is invertible and $\mathcal{R}_y(-\varphi)$ is its inverse. Moreover, $\mathrm{R}_y^{-1} = \mathrm{R}_y^T$

## Homogeneous axis rotation

- Let $\mathcal{V} = [x, y, z, w]^T$ be a homogeneous vector. The rotation of $\mathcal{V}$ by an angle $\kappa$ around $Z$ axis, denoted $\mathcal{R}_z(\kappa)$, is such as:

$$\mathcal{R}_z(\kappa)(\mathcal{V}) = \mathrm{R}_z\mathcal{V}$$

Where:

$$\mathrm{R}_z = \begin{bmatrix} \cos\kappa & -\sin\kappa & 0 & 0 \\ \sin\kappa & \cos\kappa & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Homogeneous axis rotation

- Transform Matrices

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ 0 & \sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_z = \begin{bmatrix} \cos\kappa & -\sin\kappa & 0 & 0 \\ \sin\kappa & \cos\kappa & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Inverse transform matrices

$$R_x^{-1} = R_x^{\top} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_y^{-1} = R_y^{\top} = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_z^{-1} = R_z^{\top} = \begin{bmatrix} \cos\varphi & \sin\kappa & 0 & 0 \\ -\sin\kappa & \cos\kappa & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## Homogeneous rotation

- The rotation of angles $\omega, \varphi, \kappa$ respectively around $X, Y$ and $Z$ axis, denoted $\mathcal{R}_{xyz}(\omega, \varphi, \kappa)$, is such as:

$$\mathcal{R}_{xyz}(\omega, \varphi, \kappa) \equiv \mathrm{R}_z \mathrm{R}_y \mathrm{R}_x$$

# Homogeneous rotation

- The rotation of angles $\omega, \varphi, \kappa$ respectively around $X, Y$ and $Z$ axis, denoted $\mathcal{R}_{xyz}(\omega, \varphi, \kappa)$, is such as:

$$\mathcal{R}_{xyz}(\omega, \varphi, \kappa) \equiv \mathrm{R}_z \mathrm{R}_y \mathrm{R}_x$$

$$\mathrm{R}_z \mathrm{R}_y \mathrm{R}_x = \begin{bmatrix} \cos\kappa & -\sin\kappa & 0 & 0 \\ \sin\kappa & \cos\kappa & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ 0 & \sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Homogeneous rotation

- The rotation of angles $\omega, \varphi, \kappa$ respectively around $X, Y$ and $Z$ axis, denoted $\mathcal{R}_{xyz}(\omega, \varphi, \kappa)$, is such as:

$$\mathcal{R}_{xyz}(\omega, \varphi, \kappa) \equiv \mathrm{R}_z \mathrm{R}_y \mathrm{R}_x$$

$$\mathrm{R}_z \mathrm{R}_y \mathrm{R}_x = \begin{bmatrix} \cos\kappa & -\sin\kappa & 0 & 0 \\ \sin\kappa & \cos\kappa & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ 0 & \sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{R}_z \mathrm{R}_y \mathrm{R}_x = \begin{bmatrix} \cos\kappa & -\sin\kappa & 0 & 0 \\ \sin\kappa & \cos\kappa & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi\sin\omega & \sin\varphi\cos\omega & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ -\sin\varphi & \cos\varphi\sin\omega & \cos\varphi\cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Homogeneous rotation

- The rotation of angles $\omega, \varphi, \kappa$ respectively around $X, Y$ and $Z$ axis, denoted $\mathcal{R}_{xyz}(\omega, \varphi, \kappa)$, is such as:

$$\mathcal{R}_{xyz}(\omega, \varphi, \kappa) \equiv \mathrm{R}_z \mathrm{R}_y \mathrm{R}_x$$

$$\mathrm{R}_z \mathrm{R}_y \mathrm{R}_x = \begin{bmatrix} \cos\kappa & -\sin\kappa & 0 & 0 \\ \sin\kappa & \cos\kappa & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi\sin\omega & 0 & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ -\sin\varphi & \cos\varphi\sin\omega & \cos\varphi\cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{R}_z \mathrm{R}_y \mathrm{R}_x = \begin{bmatrix} \cos\kappa\cos\varphi & \cos\kappa\sin\varphi\sin\omega - \sin\kappa\cos\omega & \cos\kappa\sin\varphi\cos\omega + \sin\kappa\sin\omega & 0 \\ \sin\kappa\cos\varphi & \sin\kappa\sin\varphi\sin\omega + \cos\kappa\cos\omega & \sin\kappa\sin\varphi\cos\omega - \cos\kappa\sin\omega & 0 \\ -\sin\varphi & \cos\varphi\sin\omega & \cos\varphi\cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## Homogeneous transform combination

- Homogeneous transforms combination is obtained by multiplying corresponding matrices.

- **Definition:** Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be transforms respectively represented by homogeneous $4 \times 4$ matrices $\mathrm{F}_1, \dots, \mathrm{F}_n$. The **composed transform** resulting from the composition of $\mathcal{F}_i$ transforms, defined by $\mathcal{G} = \mathcal{F}_1 \circ \cdots \circ \mathcal{F}_n$, is represented by the $\mathrm{G}$ matrix such as:

$$\mathrm{G} = \prod_{i=n}^{1} \mathrm{F}_i = \mathrm{F}_n \mathrm{F}_{n-1} \dots \mathrm{F}_1$$

## Rigid-body transformation

- **Definition:** A rigid body transformation $\mathcal{B}$ that combine translations and rotations is represented by a matrix B such as:

$$\mathrm{B} = \begin{bmatrix} & \mathrm{R} & & T \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{31} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with $\mathrm{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{31} & r_{33} \end{bmatrix}$ and $\mathrm{T} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$

# Rigid-body inverse transformation

- **Definition:** The inverse of a rigid body transformation $\mathcal{B}$ is a rigid body transformation $\mathcal{B}^{-1}$ that is represented by a matrix $\mathrm{B}^{-1}$ such as:

$$\mathrm{B}^{-1} = \begin{bmatrix} & \mathrm{R}^{-1} & & -\mathrm{R}^{-1}T \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & \mathrm{R}^{\mathsf{T}} & & -\mathrm{R}^{\mathsf{T}}\mathrm{T} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with $\mathrm{B} = \begin{bmatrix} & \mathrm{R} & & T \\ 0 & 0 & 0 & 1 \end{bmatrix}$

# Representing Transforms

| Transform | Computation | Linearity | Distances | Angles |
|---|---|---|---|---|
| Translation | $\begin{bmatrix} s_x r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & s_y r_{22} & r_{23} & t_2 \\ r_{31} & r_{31} & s_z r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ | Linear | ✓ | ✓ |
| Rotation | | Linear | ✓ | ✓ |
| Scale (uniform) | | Linear | ✗ | ✓ |
| Scale | | Linear | ✗ | ✗ |

$$\mathrm{B} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{31} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \qquad \mathrm{T} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \qquad \mathrm{R}^\top = \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

$$-\mathrm{R}^\top\mathrm{T} = \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} -r_{11}t_1 - r_{21}t_2 - r_{31}t_3 \\ -r_{12}t_1 - r_{22}t_2 - r_{32}t_3 \\ -r_{13}t_1 - r_{23}t_2 - r_{33}t_3 \end{bmatrix}$$

$$\mathrm{B}^{-1} = \begin{bmatrix} r_{11} & r_{21} & r_{31} & -r_{11}t_1 - r_{21}t_2 - r_{31}t_3 \\ r_{12} & r_{22} & r_{32} & -r_{12}t_1 - r_{22}t_2 - r_{32}t_3 \\ r_{13} & r_{23} & r_{33} & -r_{13}t_1 - r_{23}t_2 - r_{33}t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$V' = \begin{bmatrix} r_{11}v_1 + r_{12}v_2 + r_{13}v_3 + t_1 \\ r_{21}v_1 + r_{22}v_2 + r_{33}v_3 + t_2 \\ r_{31}v_1 + r_{32}v_2 + r_{33}v_3 + t_3 \\ 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{31} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11}(r_{11}v_1 + r_{12}v_2 + r_{13}v_3 + t_1) + r_{12}(r_{21}v_1 + r_{22}v_2 + r_{33}v_3 + t_2) + r_{13}(r_{31}v_1 + r_{32}v_2 + r_{33}v_3 + t_1) + t_1 \\ \\ \\ 0 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} r_{11}r_{11}v_1 + r_{11}r_{12}v_2 + r_{11}r_{13}v_3 + r_{11}t_1 + r_{12}r_{21}v_1 + r_{12}r_{22}v_2 + r_{12}r_{33}v_3 + r_{12}t_2 + r_{13}r_{31}v_1 + r_{13}r_{32}v_2 + r_{13}r_{33}v_3 + r_{13}t_3 + t_1 \\ \\ \\ 0 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} r_{11}r_{11}v_1 + r_{12}r_{21}v_1 + r_{13}r_{31}v_1 + r_{11}r_{12}v_2 + r_{12}r_{22}v_2 + r_{13}r_{32}v_2 + r_{11}r_{13}v_3 + r_{13}r_{33}v_3 + r_{12}r_{33}v_3 + t_1r_{11} + t_1 + r_{12}t_2 + r_{13}t_3 \\ \\ \\ 0 \end{bmatrix}$$

$$V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 1 \end{bmatrix} \quad BV = \begin{bmatrix} r_{11}v_1 + r_{12}v_2 + r_{13}v_3 + t_1 \\ r_{21}v_1 + r_{22}v_2 + r_{33}v_3 + t_2 \\ r_{31}v_1 + r_{32}v_2 + r_{33}v_3 + t_1 \\ 1 \end{bmatrix} = V'$$

$$B^{-1} = \begin{bmatrix} r_{11} & r_{21} & r_{31} & -r_{11}t_1 - r_{21}t_2 - r_{31}t_3 \\ r_{12} & r_{22} & r_{32} & -r_{12}t_1 - r_{22}t_2 - r_{32}t_3 \\ r_{13} & r_{23} & r_{33} & -r_{13}t_1 - r_{23}t_2 - r_{33}t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{31} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11}^2 + r_{12}^2 + r_{13}^2 & & & \\ & & & \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
\mathrm{B} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{31} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$\begin{bmatrix}1&0&0&\alpha\\0&1&0&\beta\\0&0&1&\gamma\\0&0&0&1\end{bmatrix}\begin{bmatrix}kx_a+jx_b\\ky_a+jy_b\\kz_a+jz_b\\kw_a+jw_b\end{bmatrix}\begin{bmatrix}kx_a+jx_b+\alpha(k+j)\\ky_a+jy_b+\beta(k+j)\\kz_a+jz_b+\gamma(k+j)\\kw_a+jw_b\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&\alpha\\0&1&0&\beta\\0&0&1&\gamma\\0&0&0&1\end{bmatrix}\begin{bmatrix}kx_a+jx_b\\ky_a+jy_b\\kz_a+jz_b\\kw_a+jw_b\end{bmatrix}\begin{bmatrix}kx_a+jx_b+\alpha k+\alpha j\\ky_a+jy_b+\beta k+\beta j\\kz_a+jz_b+\gamma k+\gamma j\\kw_a+jw_b\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&\alpha\\0&1&0&\beta\\0&0&1&\gamma\\0&0&0&1\end{bmatrix}\begin{bmatrix}kx_a+jx_b\\ky_a+jy_b\\kz_a+jz_b\\kw_a+jw_b\end{bmatrix}\begin{bmatrix}kx_a+\alpha k+jx_b+\alpha j\\ky_a+\beta k+jy_b+\beta j\\kz_a+\gamma k+jz_b+\gamma j\\kw_a+jw_b\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&\alpha\\0&1&0&\beta\\0&0&1&\gamma\\0&0&0&1\end{bmatrix}\begin{bmatrix}kx_a+jx_b\\ky_a+jy_b\\kz_a+jz_b\\kw_a+jw_b\end{bmatrix}\begin{bmatrix}kx_a+\alpha k\\ky_a+\beta k\\kz_a+\gamma k\\kw_a\end{bmatrix}+\begin{bmatrix}jx_b+\alpha j\\jy_b+\beta j\\jz_b+\gamma j\\jw_b\end{bmatrix}$$

$$\begin{bmatrix} kx_a + jx_b \\ ky_a + jy_b \\ kz_a + jz_b \\ kw_a + jw_b \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad k\begin{bmatrix} x_a + \alpha \\ y_a + \beta \\ z_a + \gamma \\ w_a \end{bmatrix} + j\begin{bmatrix} jx_b + \alpha \\ jy_b + \beta \\ jz_b + \gamma \\ w_b \end{bmatrix}$$

$$R_x R_x^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ 0 & \sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ 0 & \sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2\omega + \sin^2\omega & \cos\omega\sin\omega - \cos\omega\sin\omega & 0 \\ 0 & \sin\omega\cos\omega - \cos\omega\sin\omega & \sin^2\omega + \cos^2\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{31} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} r_{11}x + r_{12}y + r_{13}z + t_1 w \\ r_{21}x + r_{22}y + r_{23}z + t_2 w \\ r_{31}x + r_{32}y + r_{33}z + t_3 w \\ w \end{bmatrix}$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{31} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{11}t_1 + r_{12}t_2 + r_{13}t_3 \\ r_{21} & r_{22} & r_{23} & r_{21}t_1 + r_{22}t_2 + r_{23}t_3 \\ r_{31} & r_{32} & r_{33} & r_{31}t_1 + r_{32}t_2 + r_{33}t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{31} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
R_z R_y R_x = \begin{bmatrix} \cos\kappa & -\sin\kappa & 0 & 0 \\ \sin\kappa & \cos\kappa & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi\sin\omega & 0 & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ -\sin\varphi & \cos\varphi\sin\omega & \cos\varphi\cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
R_z R_y R_x = \begin{bmatrix} \cos\kappa & -\sin\kappa & 0 & 0 \\ \sin\kappa & \cos\kappa & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\kappa\cos\varphi & \cos\kappa\sin\varphi\sin\omega - \sin\kappa\cos\omega & \sin\kappa\sin\omega & 0 \\ \sin\kappa\cos\varphi & \sin\kappa\sin\varphi\sin\omega + \cos\kappa\cos\omega & -\sin\kappa\sin\omega & 0 \\ -\sin\varphi & \cos\varphi\sin\omega & \cos\kappa\cos\varphi\cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
R_z R_y R_x = \begin{bmatrix} \cos\kappa\cos\varphi & \sin\varphi\sin\omega & 0 & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ -\sin\varphi & \cos\varphi\sin\omega & \cos\varphi\cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ 0 & \sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{R}_z\mathrm{R}_y\mathrm{R}_x = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi\sin\omega & \sin\varphi\cos\omega & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ -\sin\varphi & \cos\varphi\sin\omega & \cos\varphi\cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(x, y, z, w) = (2x, 2y, 2z, 2w)$$

$$(x, y, z, w) = (x/w, y/w, z/w, 1) \qquad (x, y, z, w) = (\alpha x, \alpha y, \alpha z, \alpha w), \alpha \in \mathbb{R}$$

$$(x, y, z, w) = (1/x, 1/y, 1/z, 1/w)$$

$$
R_z R_y R_x = \begin{bmatrix} \cos\kappa\cos\varphi & \cos\kappa\sin\varphi\sin\omega - \sin\kappa\cos\omega & \cos\kappa\sin\varphi\cos\omega + \sin\kappa\sin\omega & 0 \\ \sin\kappa\cos\varphi & \sin\kappa\sin\varphi\sin\omega + \cos\kappa\cos\omega & \sin\kappa\sin\varphi\cos\omega - \cos\kappa\sin\omega & 0 \\ -\sin\varphi & \cos\varphi\sin\omega & \cos\varphi\cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} \cos\varphi & \sin\varphi\sin\omega & \sin\varphi\cos\omega & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ -\sin\varphi & \cos\varphi\sin\omega & \cos\varphi\cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} \cos\kappa & -\sin\kappa & 0 & 0 \\ \sin\kappa & \cos\kappa & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\kappa\cos\varphi & \cos\kappa\sin\varphi\sin\omega - \sin\kappa\cos\omega & \cos\kappa\sin\varphi\cos\omega + \sin\kappa\sin\omega & 0 \\ \sin\kappa\cos\varphi & \sin\kappa\sin\varphi\sin\omega + \cos\kappa\cos\omega & \sin\kappa\sin\varphi\cos\omega - \cos\kappa\sin\omega & 0 \\ -\sin\varphi & \cos\varphi\sin\omega & \cos\varphi\cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$H^{-1} = \begin{bmatrix} & -R & & -T \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H^{-1} = \begin{bmatrix} & -R & & -RT \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H^{-1} = \begin{bmatrix} & R^{\top} & & -R^{\top}T \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H^{-1} = \begin{bmatrix} & -R & & -RT \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H^{-1} = \begin{bmatrix} & R^{-1} & & T^{-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$