Computer Vision

Pinhole Camera

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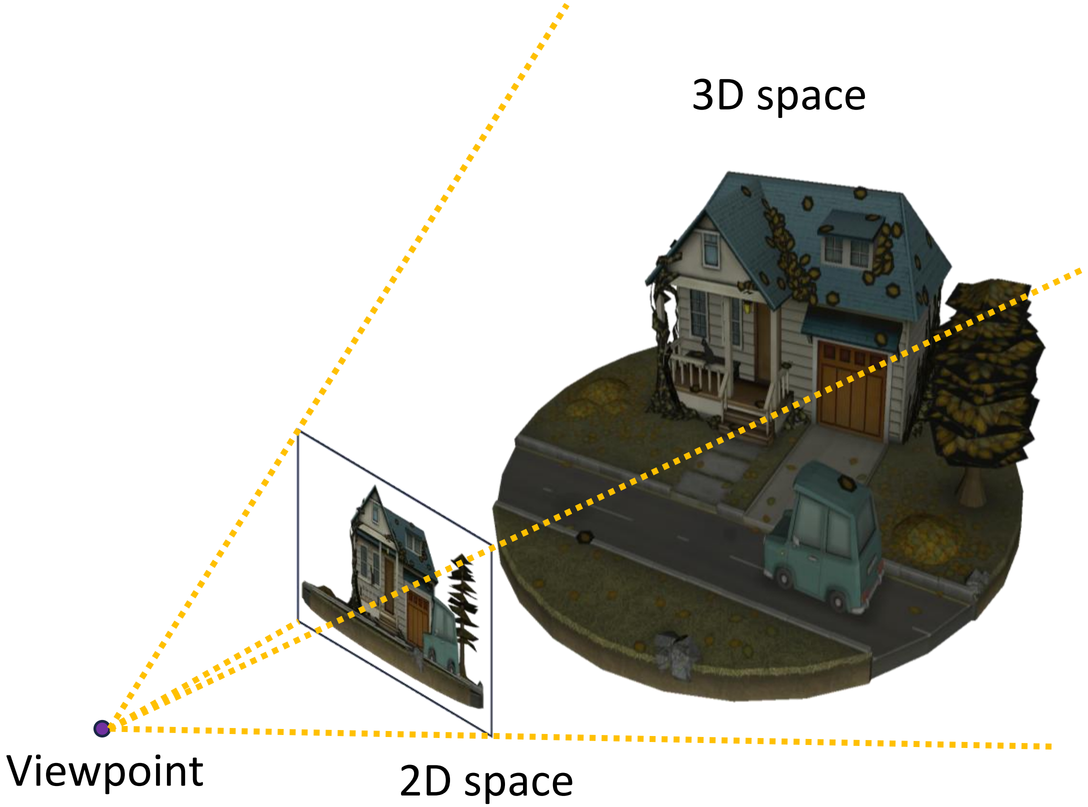
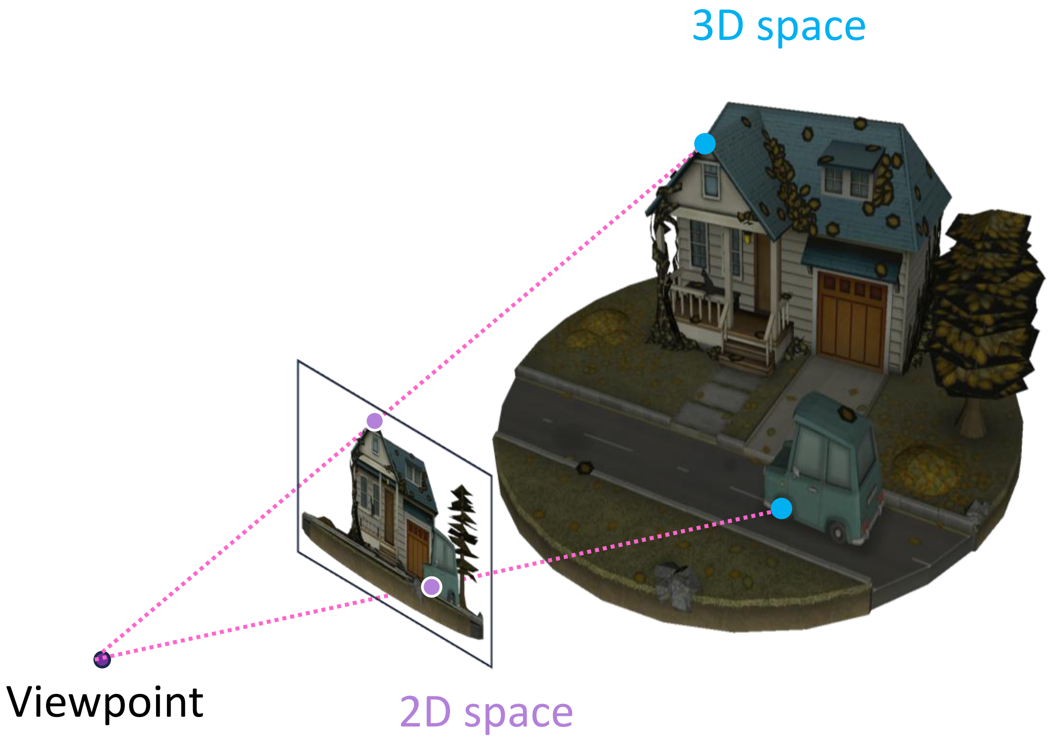
Perspective projection

- **Definition:** A perspective projection is the projection of a 3D space onto a 2D space from a viewpoint



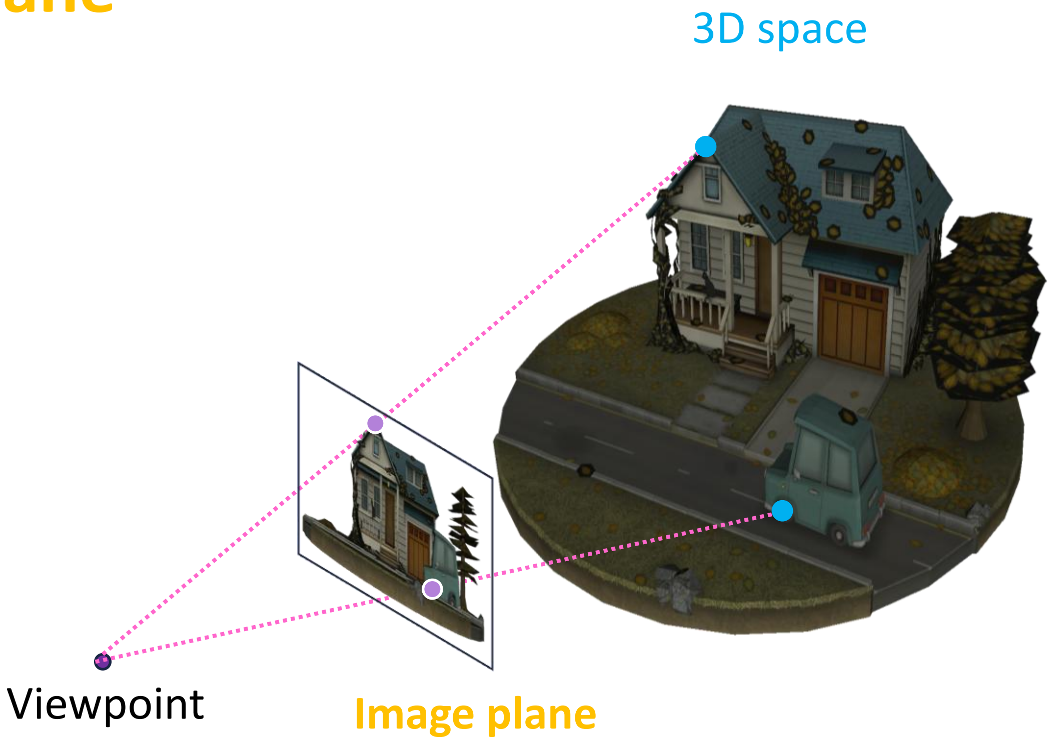
Perspective projection

- All points from the viewed 3D space can be projected onto the 2D space



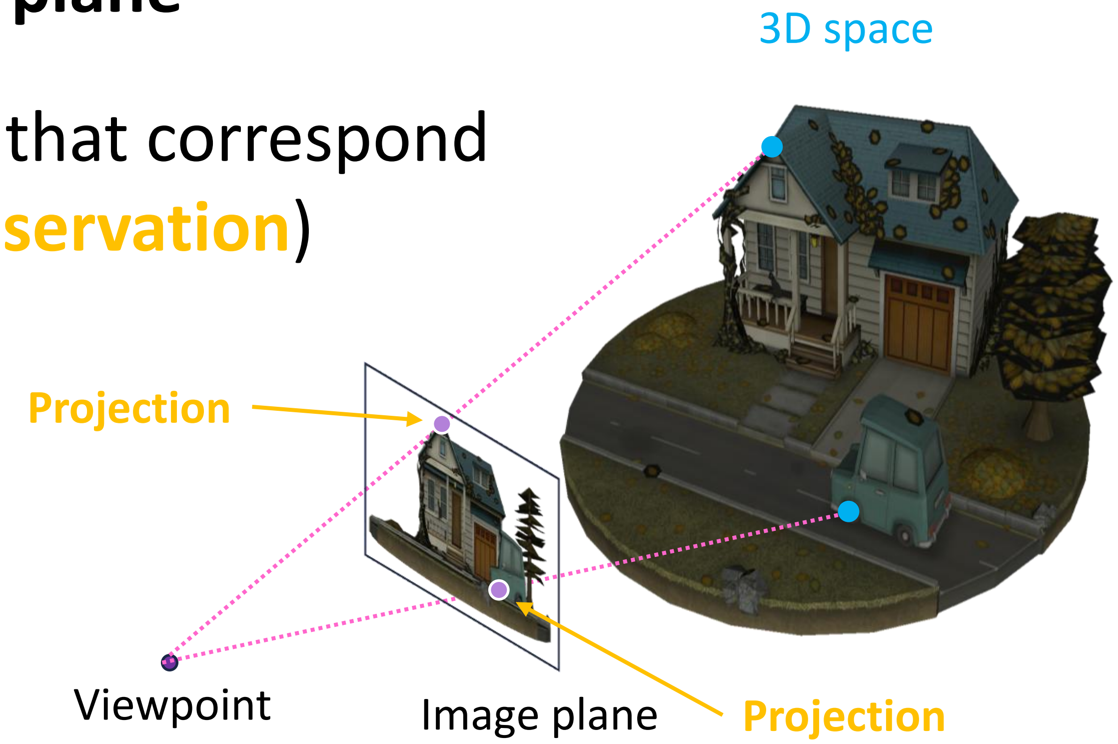
Perspective projection

- All points from the viewed 3D space can be projected onto the 2D space
- **Notation:** The 2D space is called **image plane**



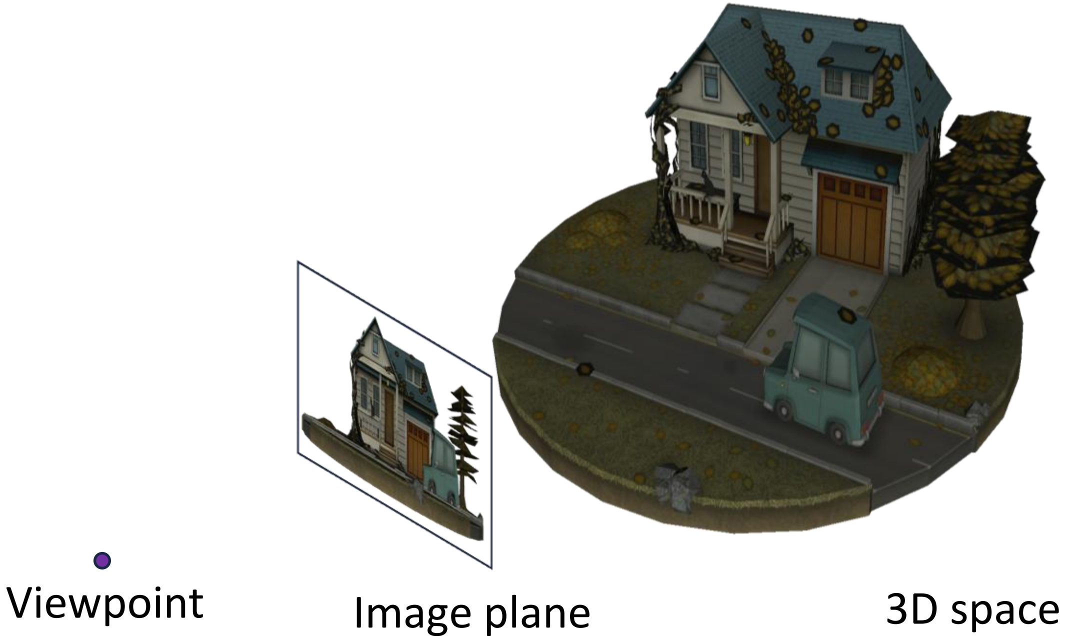
Perspective projection

- All points from the viewed 3D space can be projected onto the 2D space
- **Notation:** The 2D space is called **image plane**
- **Notation:** 2D point on the image plane that correspond to a 3D point is called **projection** (or **observation**)



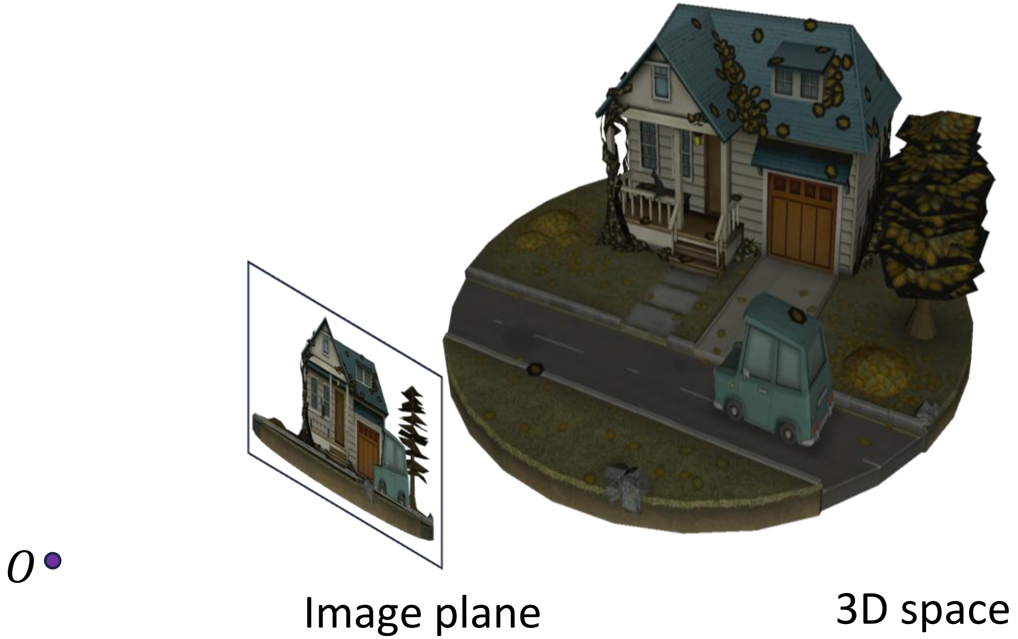
Perspective projection computation

- Let $\langle O, X_c, Y_c, Z_c \rangle$ the reference frame such as:



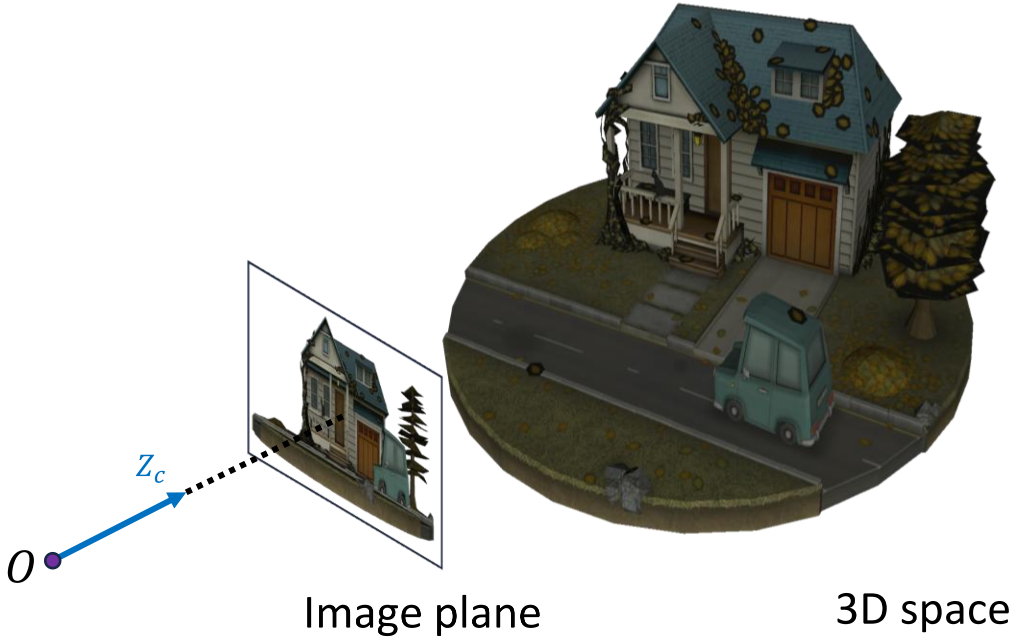
Perspective projection computation

- Let $\langle O, X_c, Y_c, Z_c \rangle$ the reference frame such as:
  - O is the viewpoint



Perspective projection computation

- Let $\langle O, X_c, Y_c, Z_c \rangle$ the reference frame such as:
  - O is the viewpoint
  - $\overrightarrow{OZ_c}$ is normal to the image plane



Perspective projection computation

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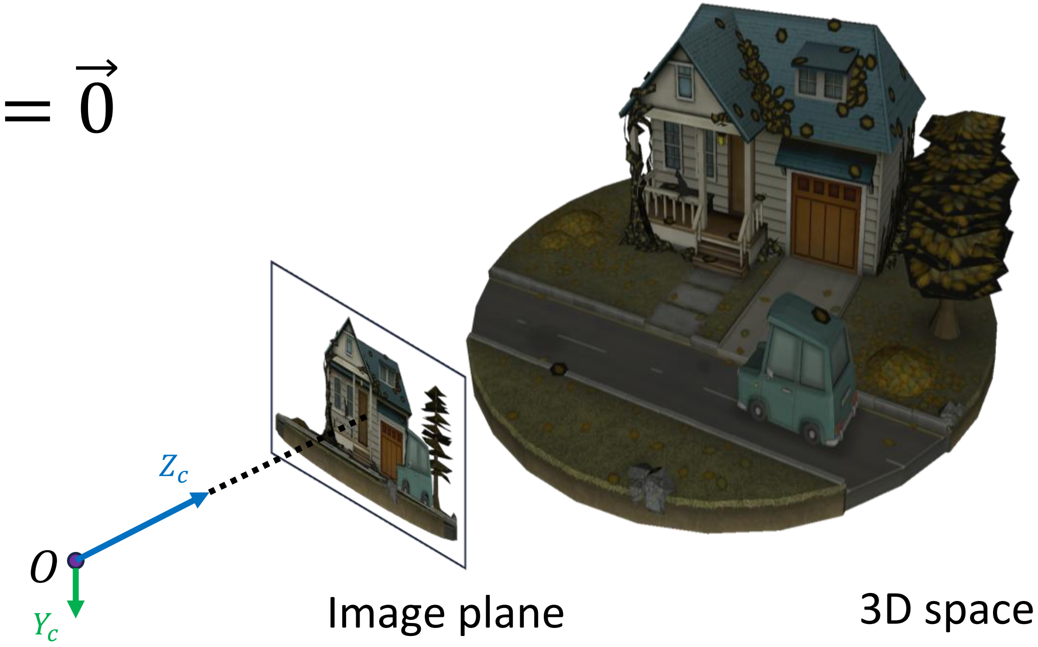
  - $\overrightarrow{OZ_c}$ is normal to the image plane
  - $\overrightarrow{OY_c}$ points downwards with $\overrightarrow{OY_c} \times \overrightarrow{OZ_c} = \vec{0}$



Perspective projection computation

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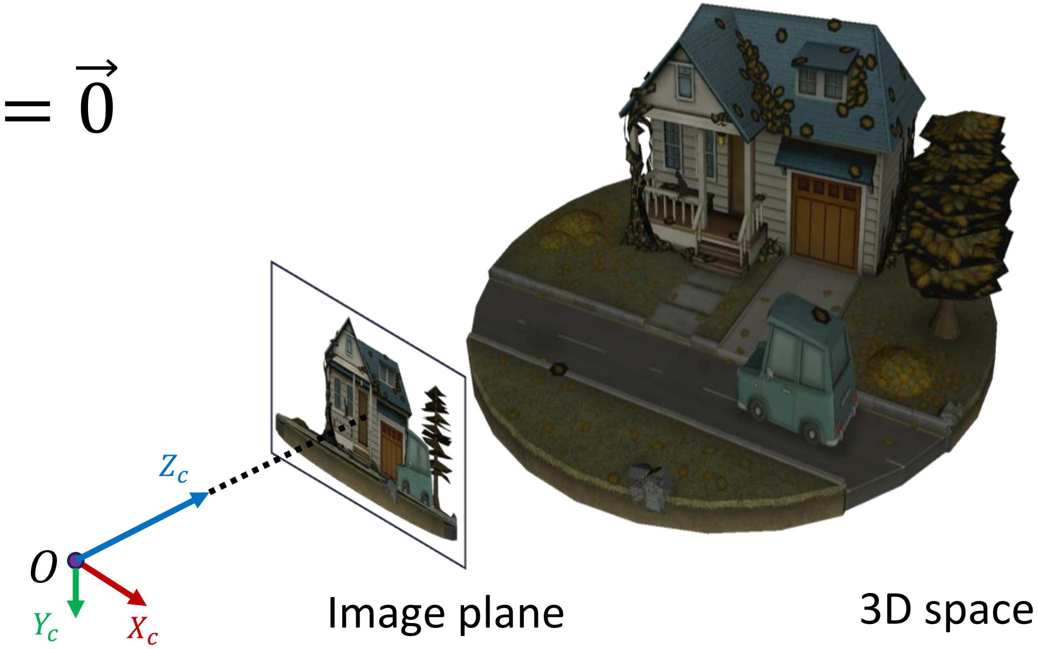
  - $\overrightarrow{OX_c} = \overrightarrow{OY_c} \times \overrightarrow{OZ_c}$



Perspective projection computation

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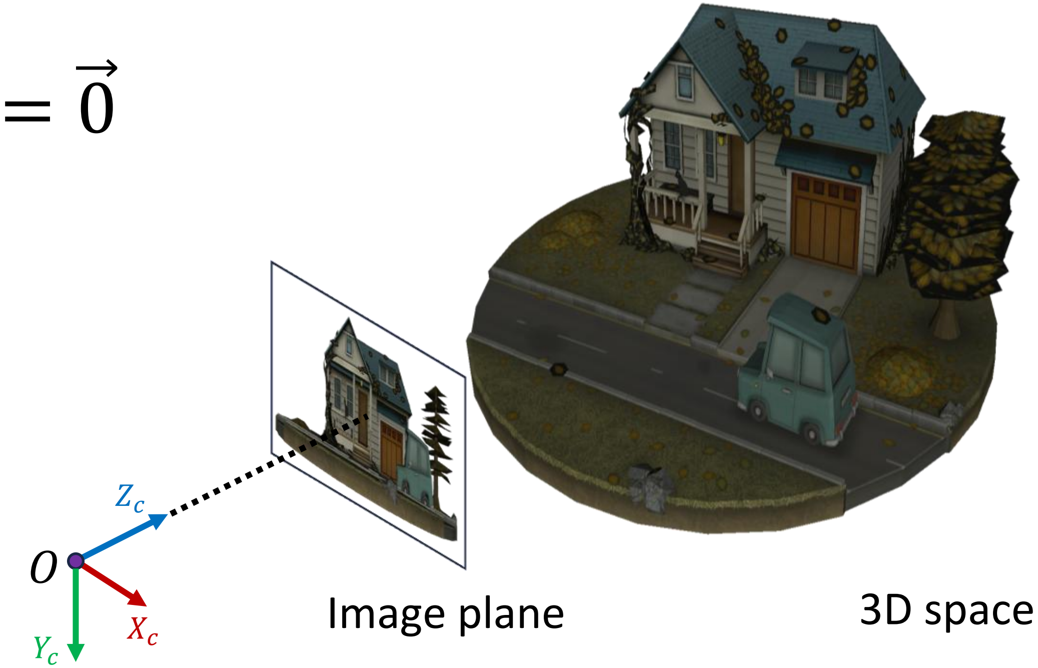
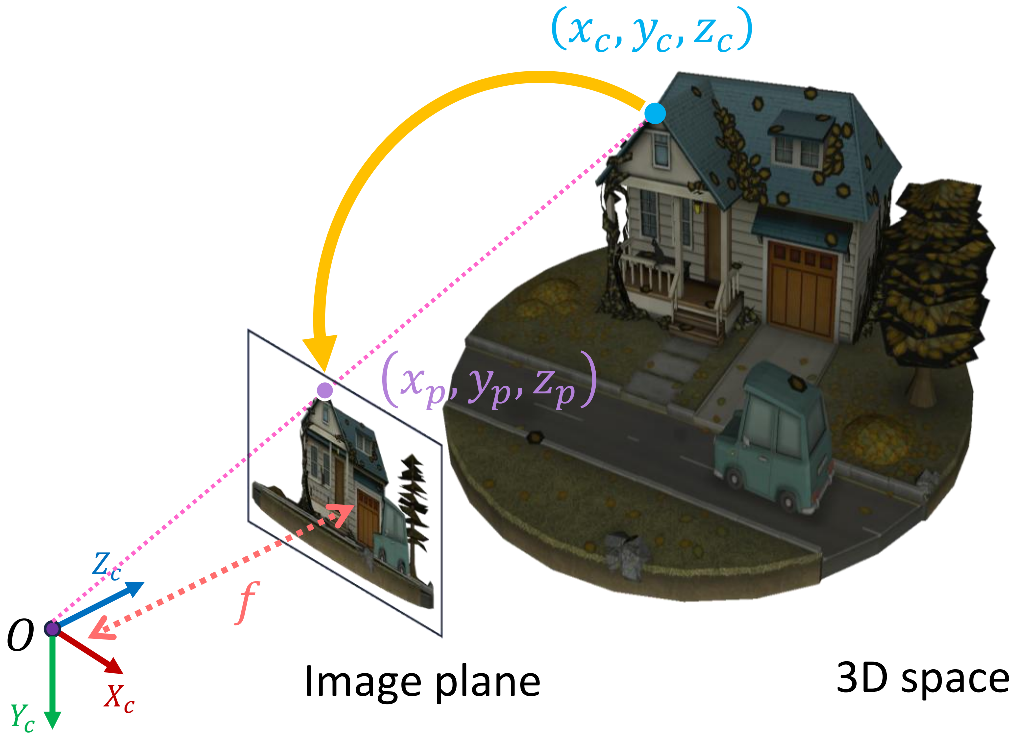
  - $\overrightarrow{OX_c} = \overrightarrow{OY_c} \times \overrightarrow{OZ_c}$
  - $\|\overrightarrow{OX_c}\| = \|\overrightarrow{OY_c}\| = \|\overrightarrow{OZ_c}\| = 1$



Perspective projection computation

- **Définition:** The Perspective projection of a 3D point (x_c, y_c, z_c) onto the image plane located at a distance f of O is a point (x_p, y_p, z_p) such as:



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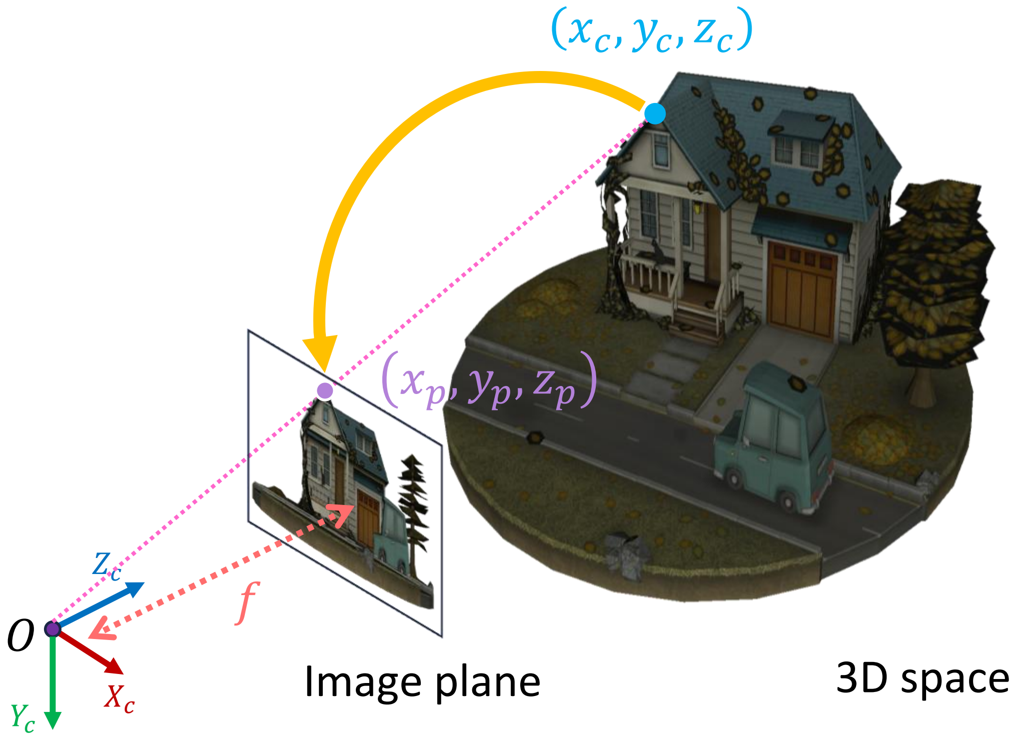
$$\begin{pmatrix} \boldsymbol{x_p} \\ \boldsymbol{y_p} \\ \boldsymbol{z_p} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f\frac{x_c}{z_c}} \\ \boldsymbol{f\frac{y}{z_c}} \\ \boldsymbol{f} \end{pmatrix}$$



Perspective projection computation

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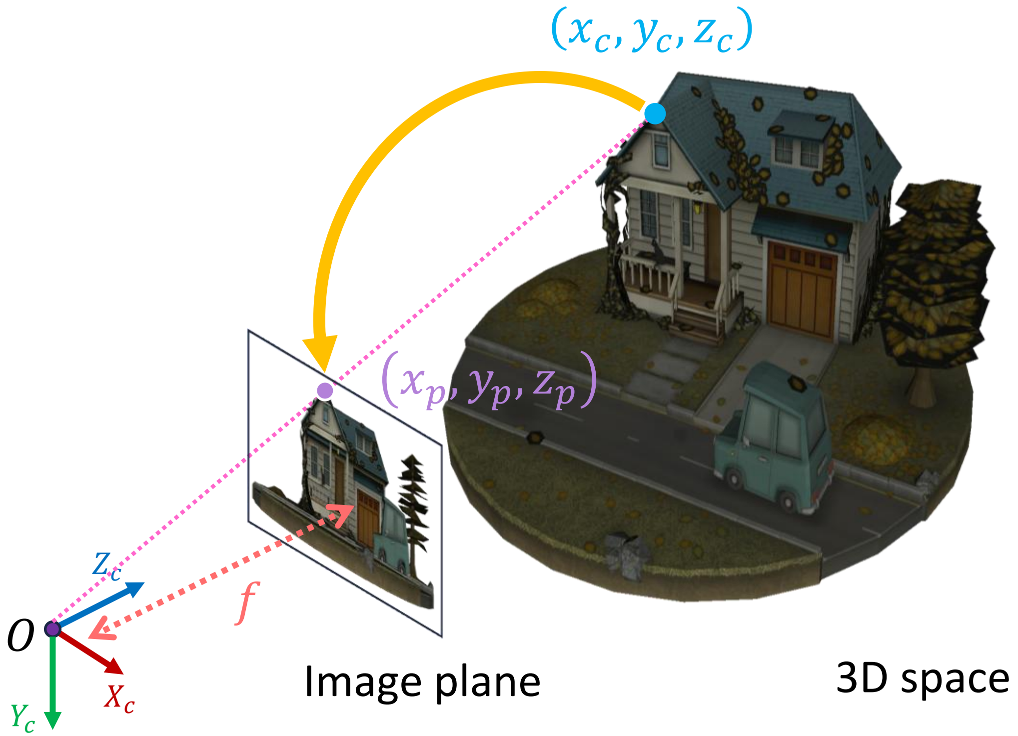
$$\alpha \begin{bmatrix} x_p \\ y_p \\ z_p \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix}$$

Homogeneous projection matrix



Perspective projection computation

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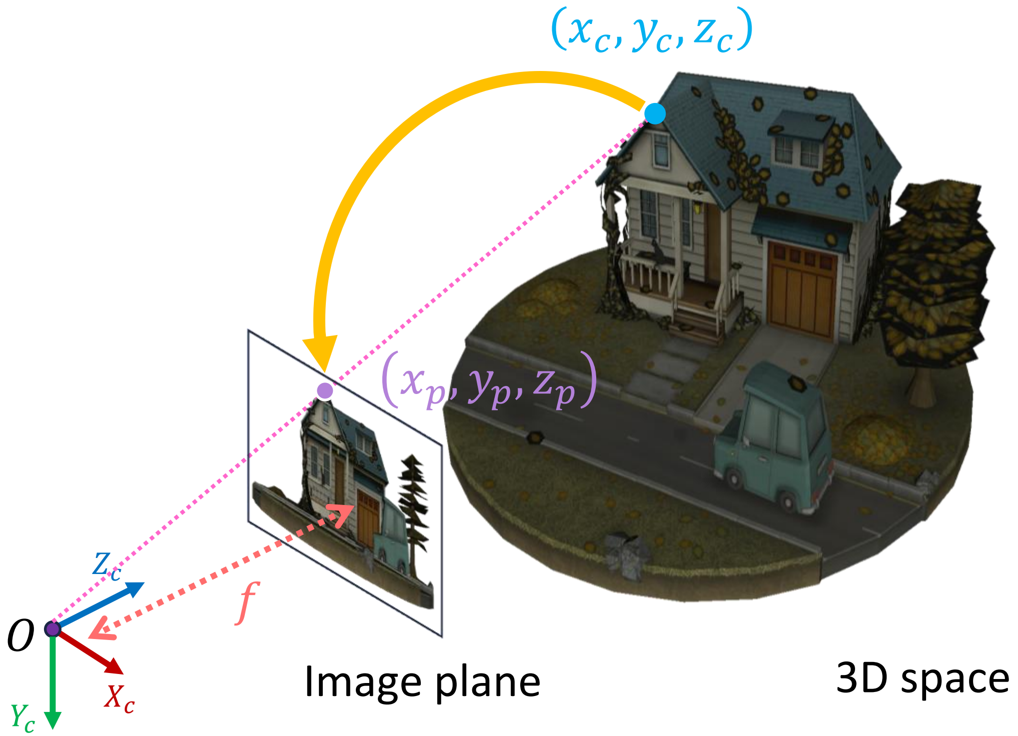

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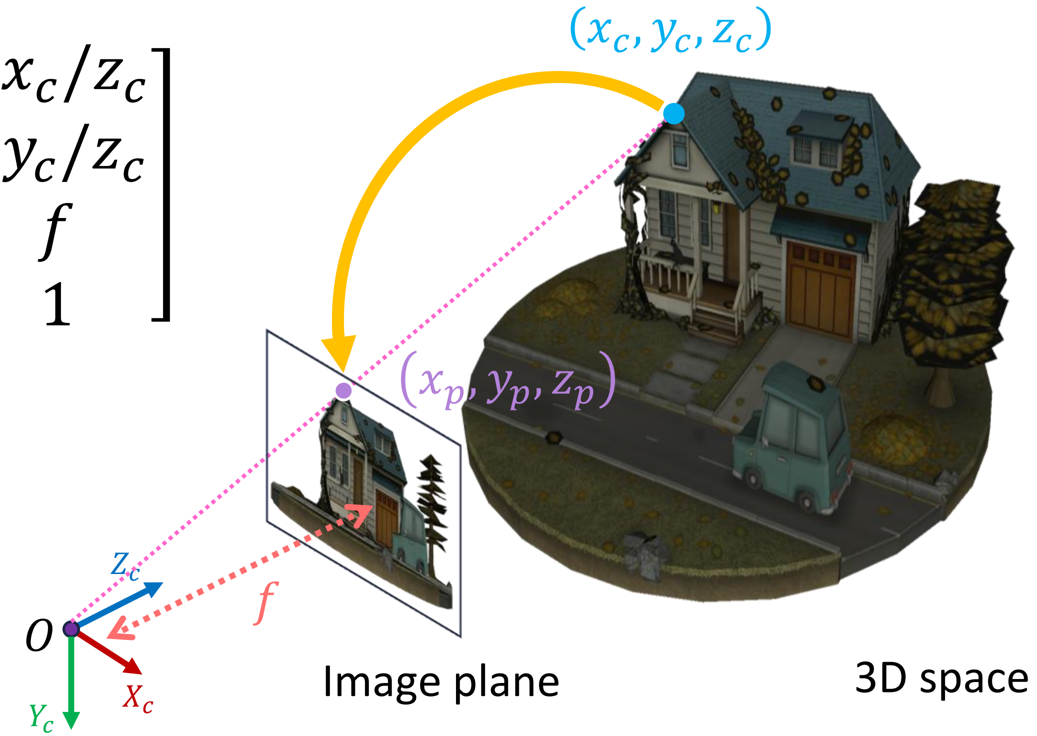

Perspective projection computation

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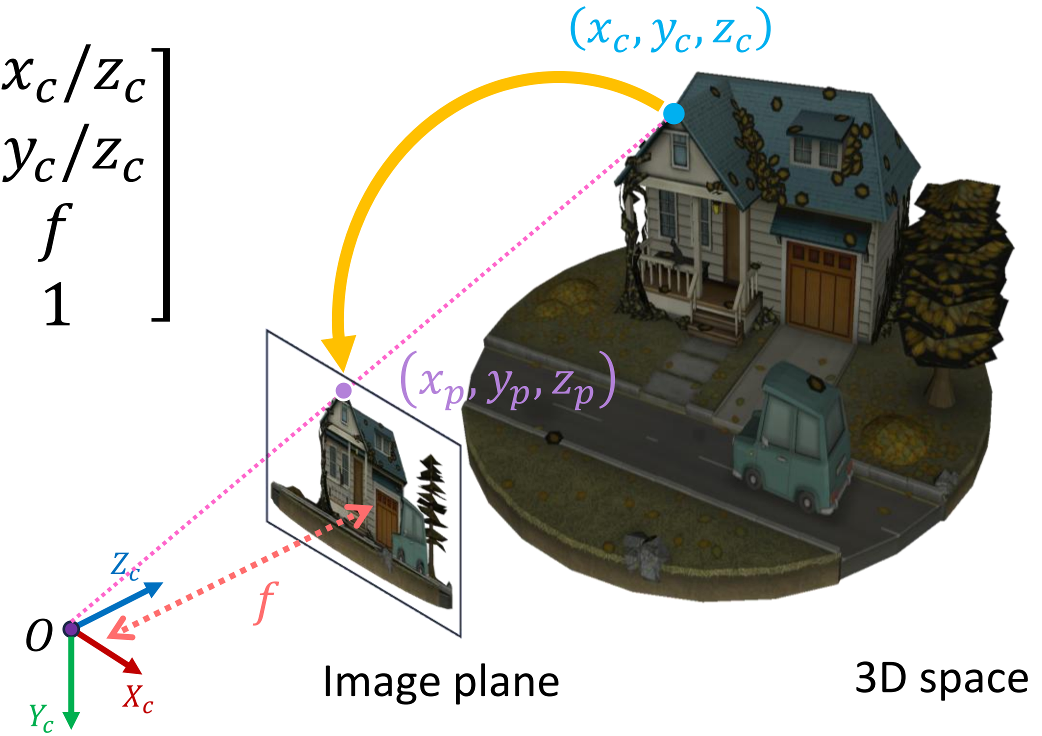
$$\alpha = \frac{Z_c}{f}$$



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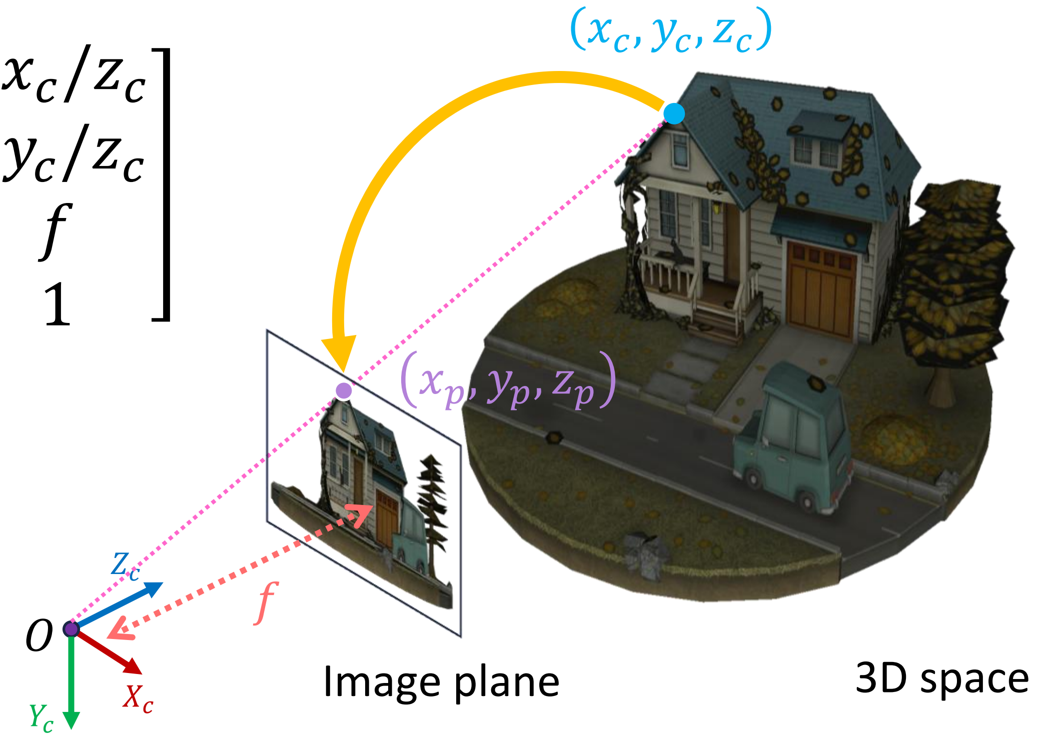
$$\alpha = \frac{Z_c}{f} \longrightarrow \begin{pmatrix} \boldsymbol{x_p} \\ \boldsymbol{y_p} \\ \boldsymbol{z_p} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f \frac{x_c}{z_c}} \\ \boldsymbol{f \frac{y}{z_c}} \\ \boldsymbol{f} \end{pmatrix}$$



Pinhole Camera

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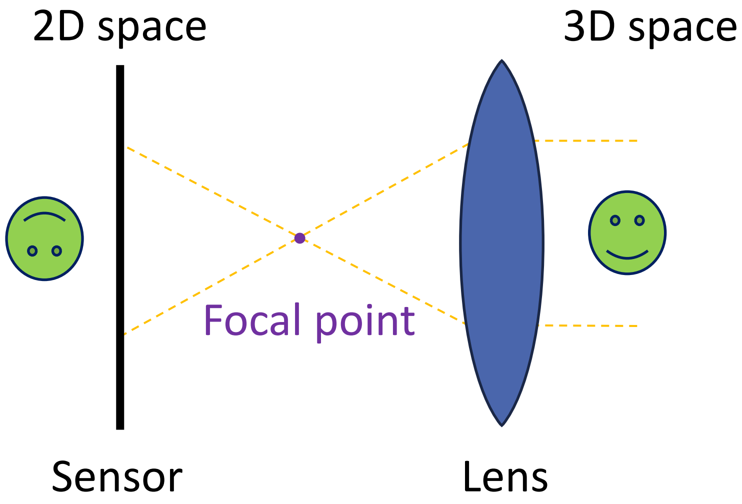
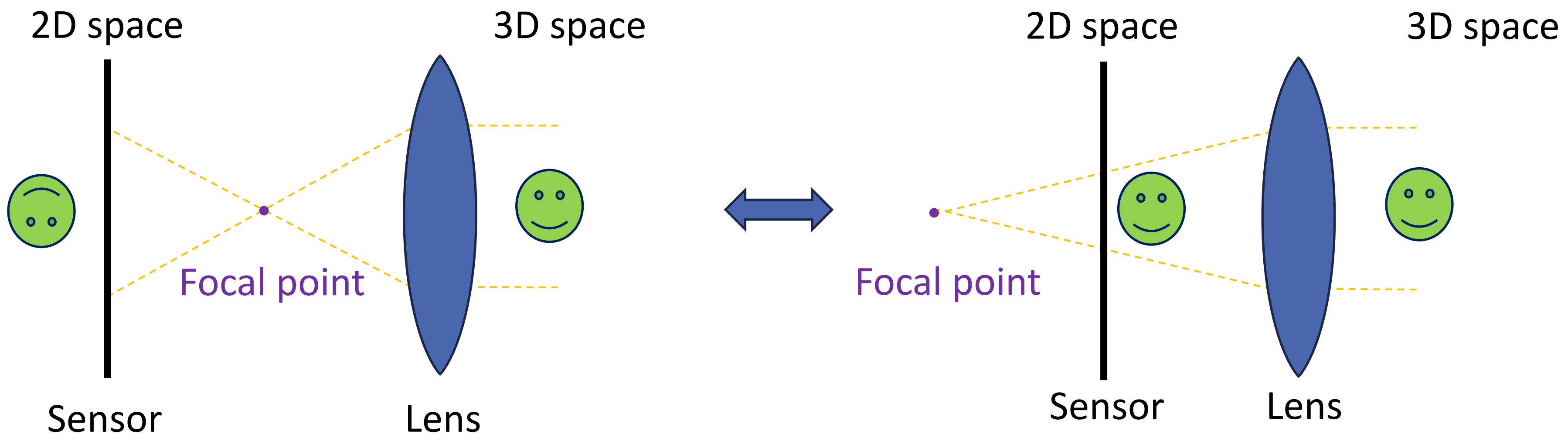


Physical representation

Photogrammetry / Definition & Principles

Pinhole Camera

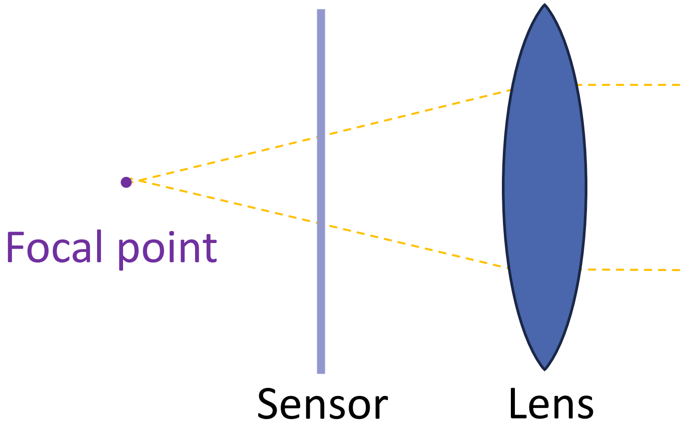
- **Definition**: A camera is a device that project a 3D space towards a frame through a lens

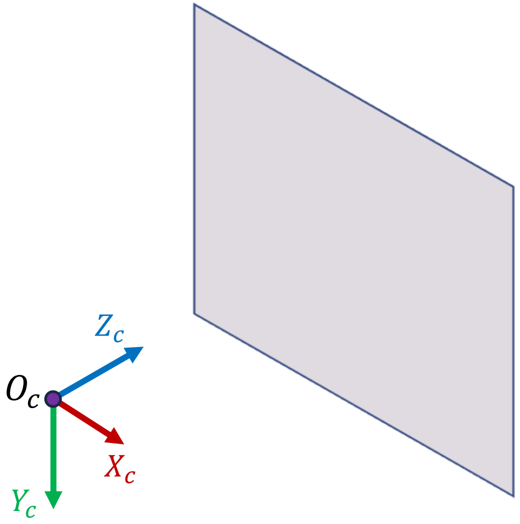


Physical representation **Virtual representation**

Pinhole Camera

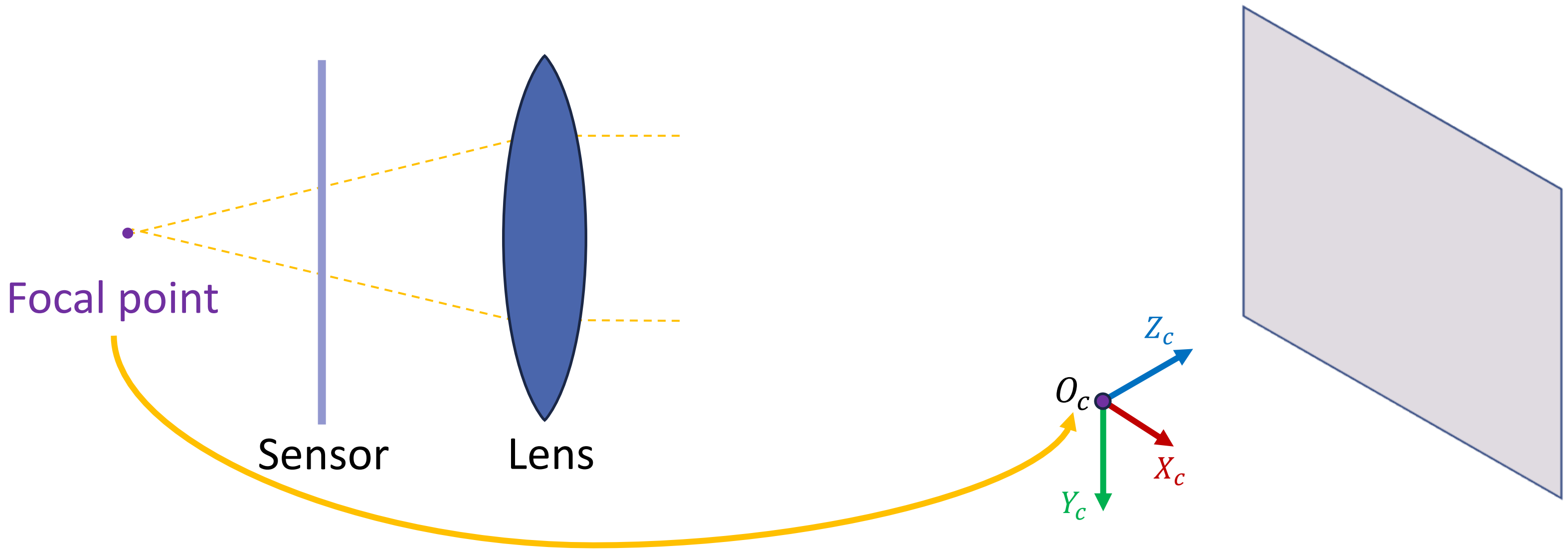
- A camera can be assimilated to a perspective projection system where:





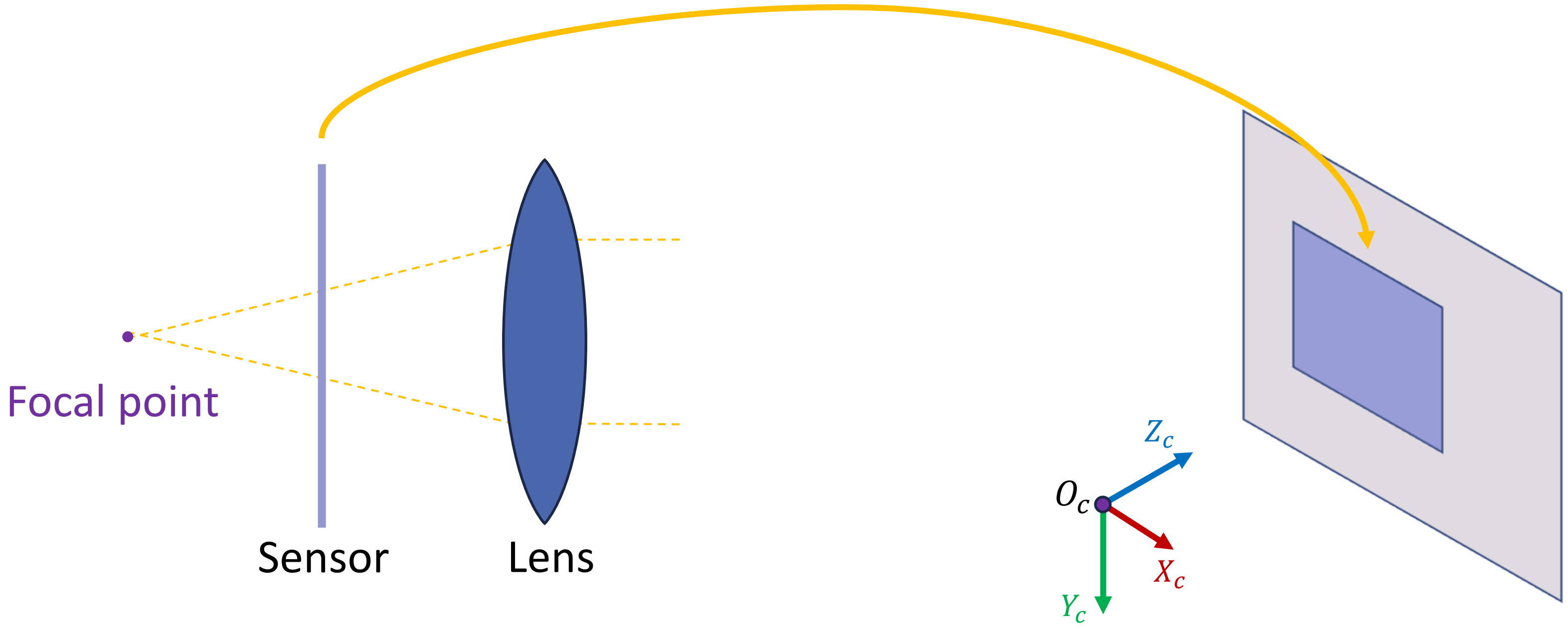
Pinhole Camera

- A camera can be assimilated to a perspective projection system where:
  - O_c is the focal point



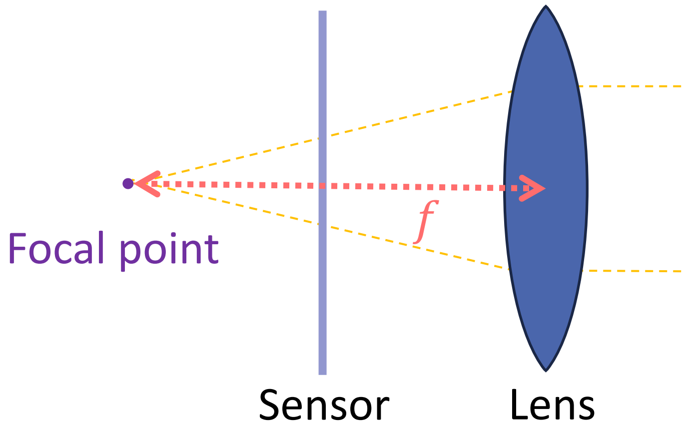
Pinhole Camera

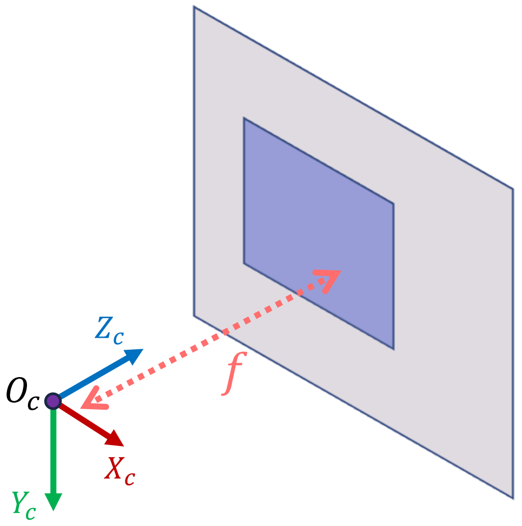
- A camera can be assimilated to a perspective projection system where:
  - O_c is the focal point
  - The sensor is located on the image plane



Pinhole Camera

- A camera can be assimilated to a perspective projection system where:
  - O_c is the focal point
  - The sensor is located on the image plane
  - f is the focal length





Photogrammetry / Definition & Principles

Pinhole Camera

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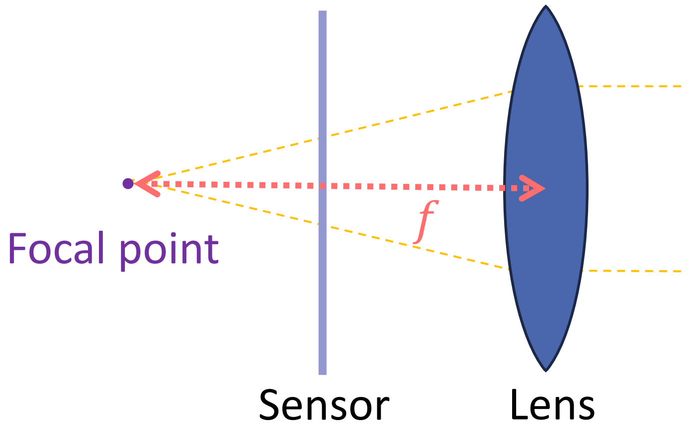
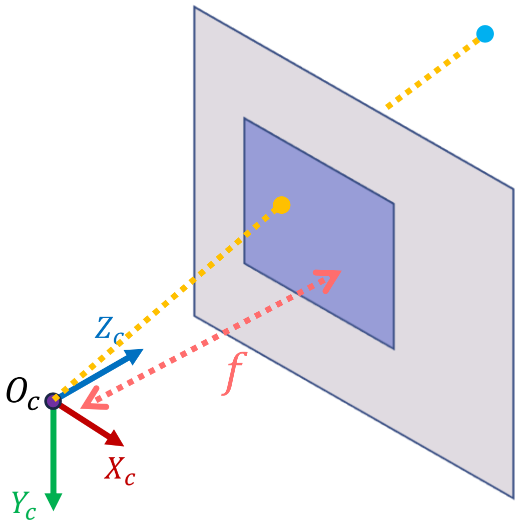
  - f is the focal length

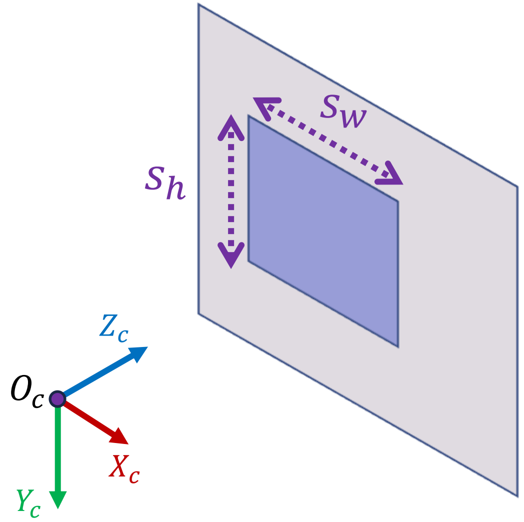
How to express a projection (x_p, y_p, z_p) according to the sensor ?





Pinhole Camera

- **Definition:** Let $\langle O_c, X_c, Y_c, Z_c \rangle$ be the camera reference frame and let be s_w and s_h the width and the length of the sensor. The sensor reference frame $\langle O_s, X_s, Y_s \rangle$ is such that:



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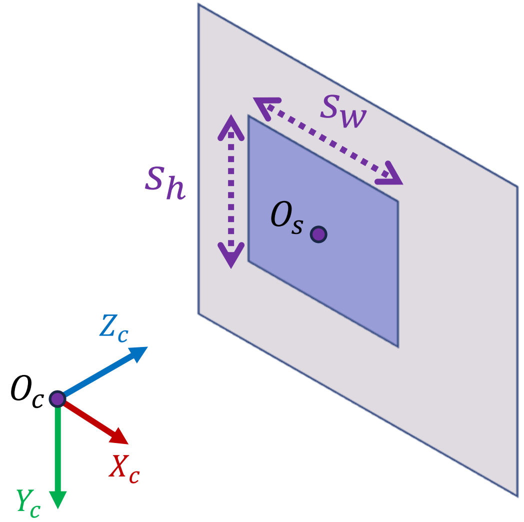

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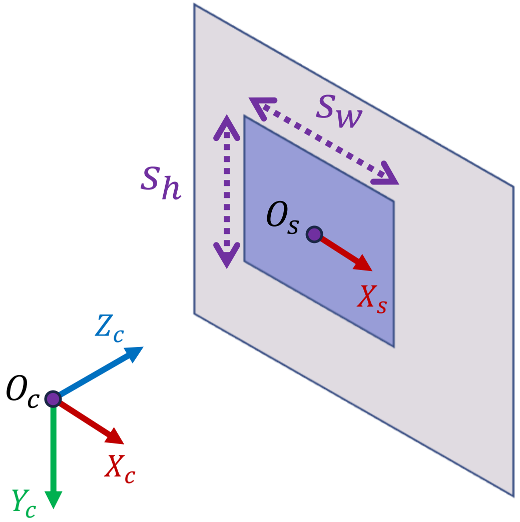
  - O_s is the center of the sensor
  - $\overrightarrow{O_s X_s}$ is colinear to $\overrightarrow{X_c}$



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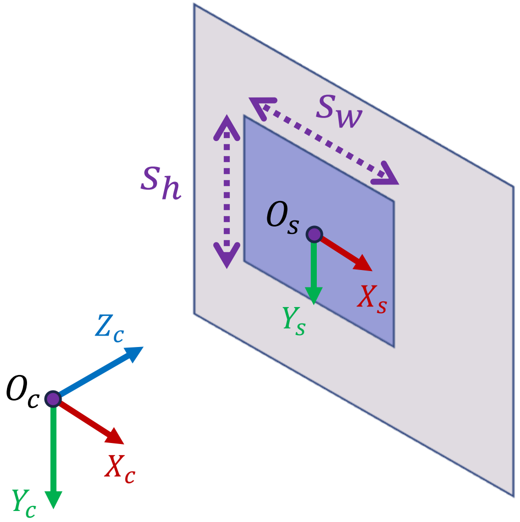

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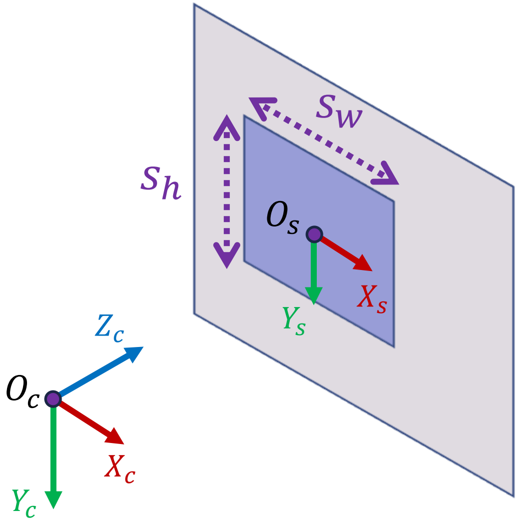
  - O_s is the center of the sensor
  - $\overrightarrow{O_s X_s}$ is colinear to $\overrightarrow{X_c}$
  - $\overrightarrow{O_s Y_s}$ is colinear to $\overrightarrow{Y_c}$
  - $\|\overrightarrow{O_s X_s}\| = \frac{1}{s_w}$
  - $\|\overrightarrow{O_s Y_s}\| = \frac{1}{s_h}$

$$\left.\begin{array}{l}\|\overrightarrow{O_s X_s}\| = \frac{1}{s_w} \\ \|\overrightarrow{O_s Y_s}\| = \frac{1}{s_h}\end{array}\right\} \text{Normalization}$$



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  - O_s is the center of the sensor
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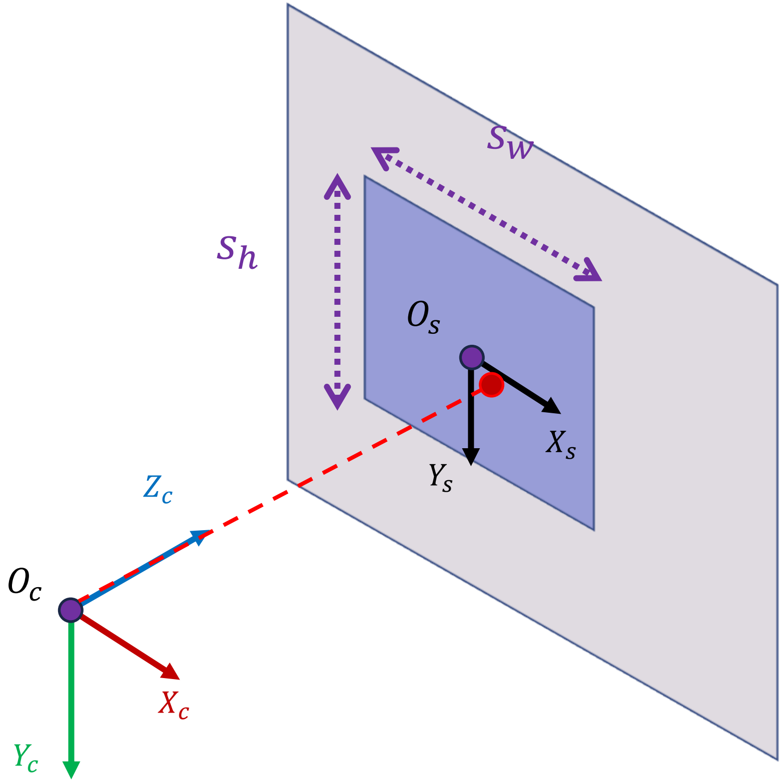
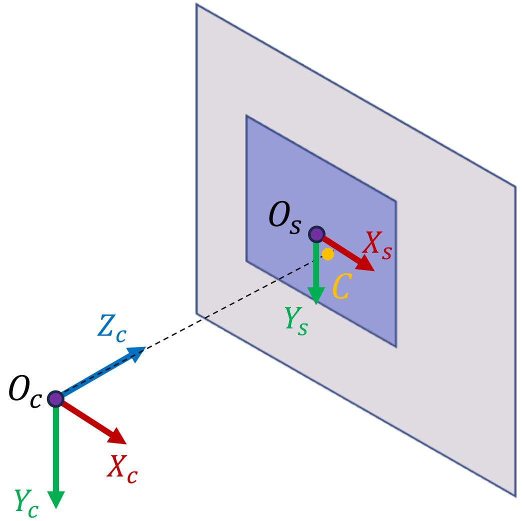
  - $\overrightarrow{O_s Y_s}$ is colinear to $\overrightarrow{Y_c}$
  - $\|\overrightarrow{O_s X_s}\| = \frac{1}{s_w}$
  - $\|\overrightarrow{O_s Y_s}\| = \frac{1}{s_h}$



O_s is not defined as the orthogonal projection of O_c

Pinhole Camera

- **Definition:** Let $\langle O_c, X_c, Y_c, Z_c \rangle$ the camera reference frame. The orthogonal projection of the focal point O_c onto the sensor is called **Principal Point** and is denoted $\boldsymbol{C} = (\boldsymbol{c_x}, \boldsymbol{c_y})$



Pinhole Camera

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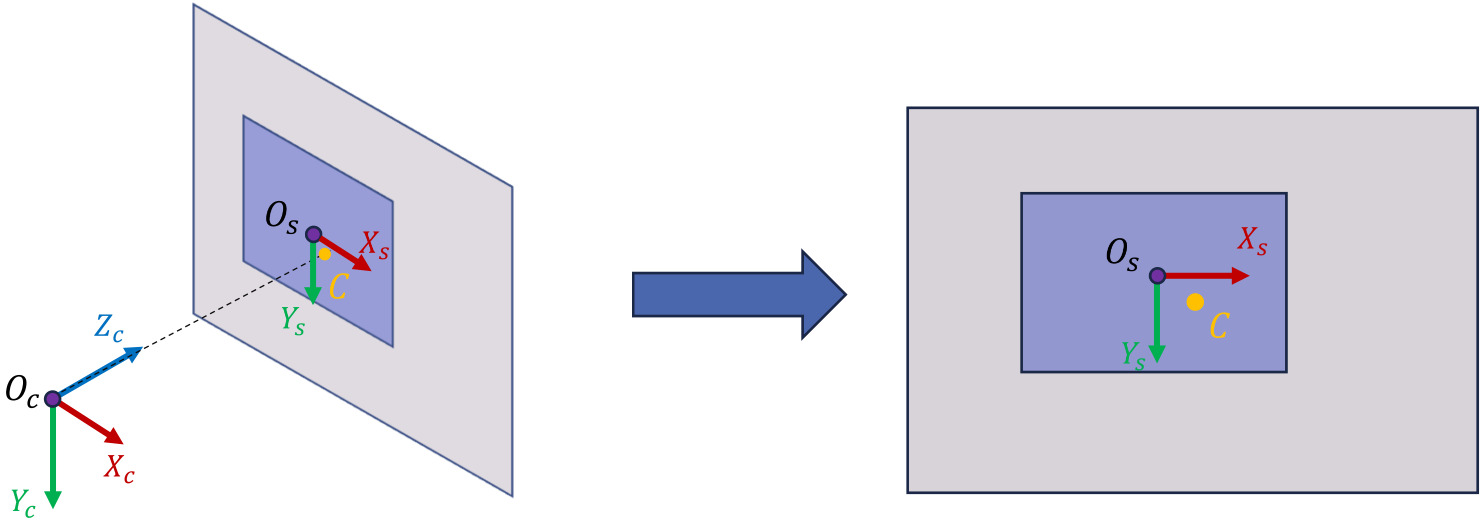
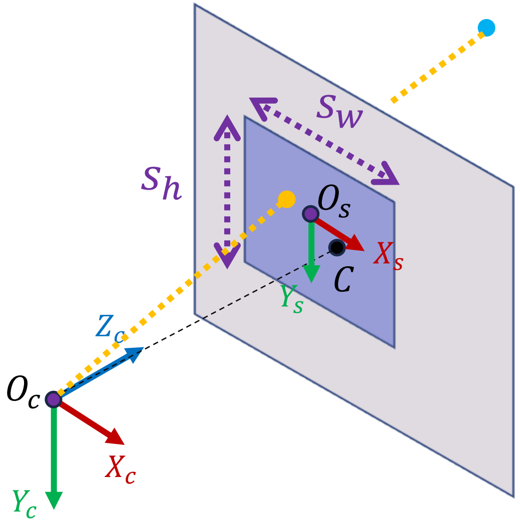
- Due to mechanical misalignment, C and O_s may differ



Pinhole Camera

- Let (x_p, y_p, z_p) the projection of a 3D point (x_c, y_c, z_c) onto the image plane expressed within the camera reference frame. The coordinates of (x_p, y_p, z_p) within the sensor reference frame, denoted (x_s, y_s), are such that:

$$\begin{pmatrix} x_s \\ y_s \end{pmatrix} = \begin{pmatrix} \frac{x_p}{s_w} + c_x \\ \frac{y_p}{s_h} + c_y \end{pmatrix}$$



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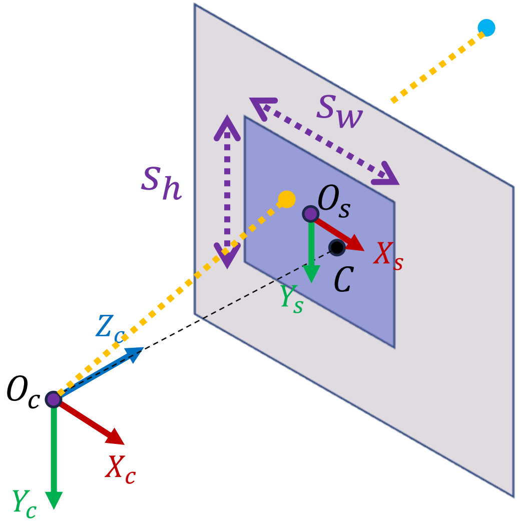
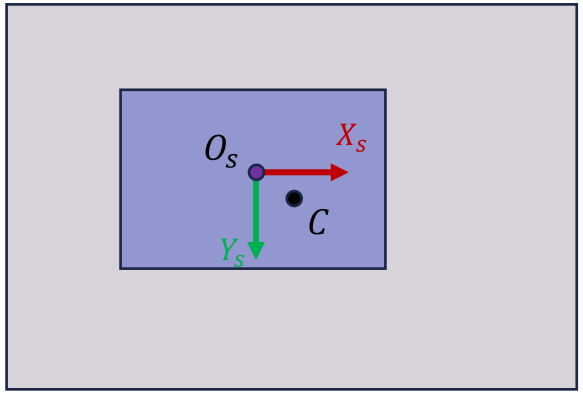
$$\begin{pmatrix} x_s \\ y_s \end{pmatrix} = \begin{pmatrix} \frac{x_p}{s_w} + c_x \\ \frac{y_p}{s_h} + c_y \end{pmatrix} = \begin{pmatrix} \frac{f x_c}{s_w z_c} + c_x \\ \frac{f y_c}{s_h z_c} + c_y \end{pmatrix}$$



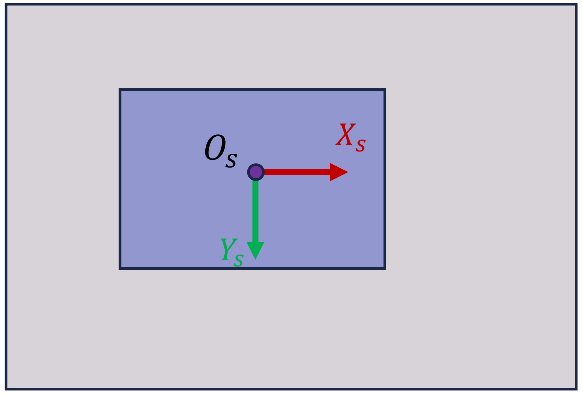
Pinhole Camera

- Digital cameras are producing raster images from their sensor
- Passing from points expressed within the sensor frame reference to pixels of the produced images is obtained by reference frame change



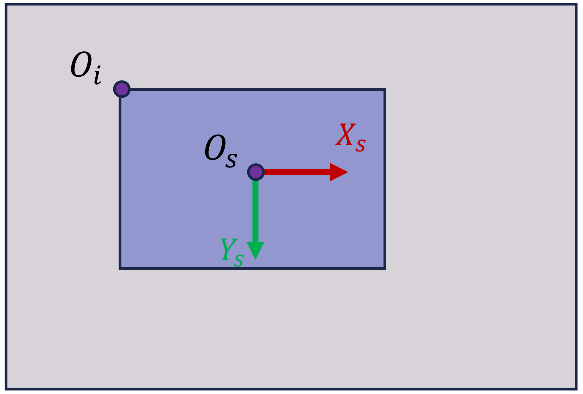
Pinhole Camera

- **Definition:** Let $\langle O_s, X_s, Y_s \rangle$ be the sensor reference frame. The image reference frame $\langle O_i, u, v \rangle$ is such that:



Pinhole Camera

- **Definition:** Let $\langle O_s, X_s, Y_s \rangle$ be the sensor reference frame. The image reference frame $\langle O_i, u, v \rangle$ is such that:
  - O_i is the top left corner of the sensor



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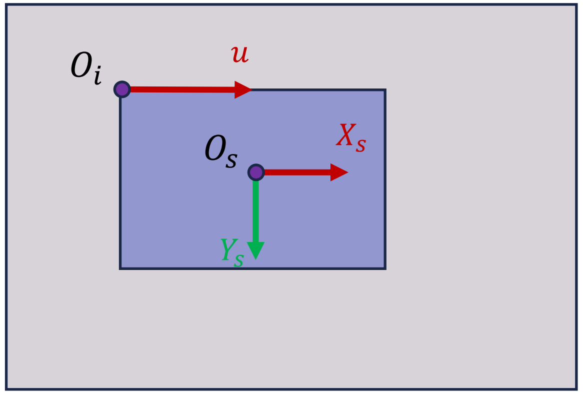
  - O_i is the top left corner of the sensor
  - $\overrightarrow{O_i u}$ is colinear to $\overrightarrow{X_s}$



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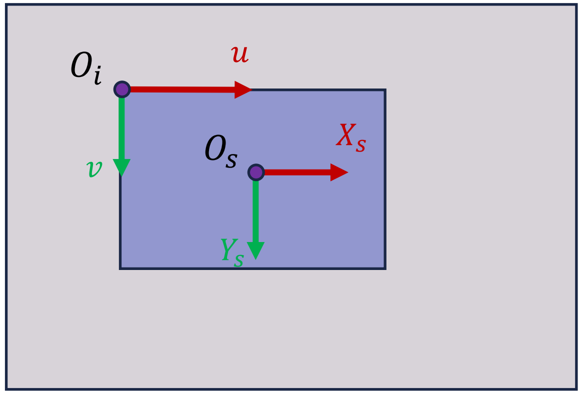
  - $\overrightarrow{O_i v}$ is colinear to $\overrightarrow{Y_s}$



Pinhole Camera

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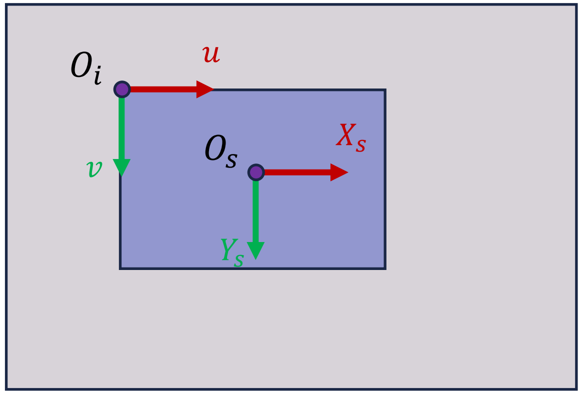
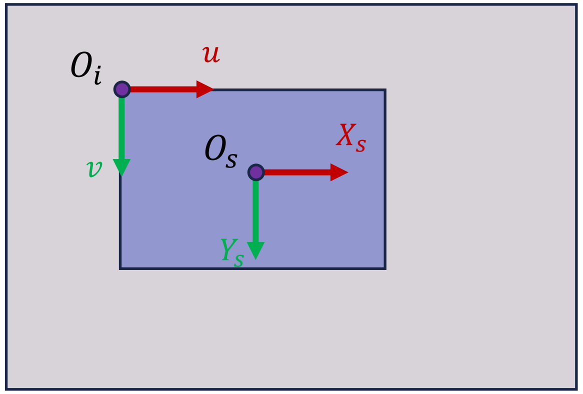
  - O_i is the top left corner of the sensor
  - $\overrightarrow{O_i u}$ is colinear to $\overrightarrow{X_s}$
  - $\overrightarrow{O_i v}$ is colinear to $\overrightarrow{Y_s}$
  - $\|\overrightarrow{O_i u}\| = \|\overrightarrow{O_i v}\| = 1\ px$



Pinhole Camera

- Let (x_s, y_s) the projection of a 3D point (x_c, y_c, z_c) onto the sensor and let i_w and i_h the width and the height of the produced image. The coordinates of (x_s, y_s) within the image reference frame, denoted (u, v), are such that:

$$\binom{u}{v} = \begin{pmatrix} i_w x_s + \frac{1}{2} i_w \\ i_h y_s + \frac{1}{2} i_h \end{pmatrix}$$



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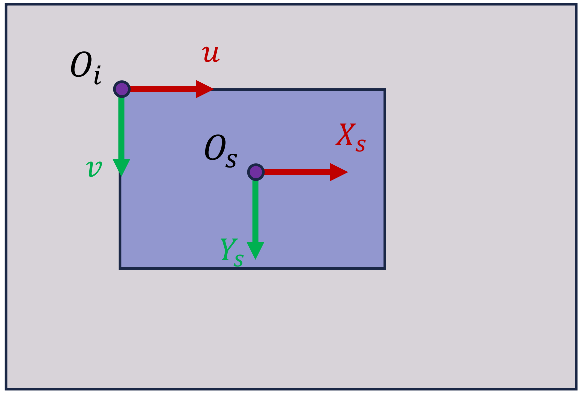
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} i_w x_s + \frac{1}{2} i_w \\ i_h y_s + \frac{1}{2} i_h \end{pmatrix}$$

Coordinates of O_s within $\langle O_i, u, v\rangle$



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$$\binom{u}{v} = \begin{pmatrix} i_w x_s + \frac{1}{2} i_w \\ i_h y_s + \frac{1}{2} i_h \end{pmatrix} = \begin{pmatrix} i_w \left(\frac{f x_c}{s_w z_c} + c_x \right) + \frac{1}{2} i_w \\ i_h \left(\frac{f y_c}{s_h z_c} + c_y \right) + \frac{1}{2} i_h \end{pmatrix}$$

Pinhole Camera

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Pinhole Camera

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$$\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix} i_w x_s + \frac{1}{2} i_w \\ i_h y_s + \frac{1}{2} i_h \end{pmatrix} = \begin{pmatrix} i_w\left(\frac{f x_c}{s_w z_c} + c_x\right) + \frac{1}{2} i_w \\ i_h\left(\frac{f y_c}{s_h z_c} + c_y\right) + \frac{1}{2} i_h \end{pmatrix} = \begin{pmatrix} \frac{i_w f}{s_w}\frac{x_c}{z_c} + i_w c_x + \frac{1}{2} i_w \\ \frac{i_h f}{s_h}\frac{y_c}{z_c} + i_h c_y + \frac{1}{2} i_h \end{pmatrix} = \begin{pmatrix} \frac{i_w f}{s_w}\frac{x_c}{z_c} + i_w\left(c_x + \frac{1}{2}\right) \\ \frac{i_h f}{s_h}\frac{y_c}{z_c} + i_h\left(c_y + \frac{1}{2}\right) \end{pmatrix}$$

Pinhole Camera

- Let (x_s, y_s) the projection of a 3D point (x_c, y_c, z_c) onto the sensor and let i_w and i_h the width and the height of the produced image. The coordinates of (x_s, y_s) within the image reference frame, denoted (u, v), are such that:

$$\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} \dfrac{\boldsymbol{i_w f}}{\boldsymbol{s_w}} \dfrac{\boldsymbol{x_c}}{\boldsymbol{z_c}} + \boldsymbol{i_w}\left(\boldsymbol{c_x} + \dfrac{\mathbf{1}}{\mathbf{2}}\right) \\ \dfrac{\boldsymbol{i_h f}}{\boldsymbol{s_h}} \dfrac{\boldsymbol{y_c}}{\boldsymbol{z_c}} + \boldsymbol{i_h}\left(\boldsymbol{c_y} + \dfrac{\mathbf{1}}{\mathbf{2}}\right) \end{pmatrix}$$

Pinhole Camera

- Let (x_s, y_s) the projection of a 3D point (x_c, y_c, z_c) onto the sensor and let i_w and i_h the width and the height of the produced image. The coordinates of (x_s, y_s) within the image reference frame, denoted (u, v), are such that:

$$\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f_u} \dfrac{\boldsymbol{x_c}}{\boldsymbol{z_c}} + \boldsymbol{i_w}\left(\boldsymbol{c_x} + \dfrac{\mathbf{1}}{\mathbf{2}}\right) \\ \dfrac{\boldsymbol{i_h f}}{\boldsymbol{s_h}} \dfrac{\boldsymbol{y_c}}{\boldsymbol{z_c}} + \boldsymbol{i_h}\left(\boldsymbol{c_y} + \dfrac{\mathbf{1}}{\mathbf{2}}\right) \end{pmatrix}$$

With:

- $f_u = \frac{i_w}{s_w} f$

Pinhole Camera

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$$\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f_u \frac{x_c}{z_c} + i_w \left(c_x + \frac{1}{2}\right)} \\ \boldsymbol{f_v \frac{y_c}{z_c} + i_h \left(c_y + \frac{1}{2}\right)} \end{pmatrix}$$

With:

- $f_u = \frac{i_w}{s_w} f$ and $f_v = \frac{i_h}{s_h} f$

Pinhole Camera

- Let (x_s, y_s) the projection of a 3D point (x_c, y_c, z_c) onto the sensor and let i_w and i_h the width and the height of the produced image. The coordinates of (x_s, y_s) within the image reference frame, denoted (u, v), are such that:

$$\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f_u \frac{x_c}{z_c} + c_u} \\ \boldsymbol{f_v \frac{y_c}{z_c} + i_h \left(c_y + \frac{1}{2} \right)} \end{pmatrix}$$

With:

- $f_u = \frac{i_w}{s_w} f$ and $f_v = \frac{i_h}{s_h} f$
- $c_u = i_w \left(c_x + \frac{1}{2} \right)$

Pinhole Camera

- Let (x_s, y_s) the projection of a 3D point (x_c, y_c, z_c) onto the sensor and let i_w and i_h the width and the height of the produced image. The coordinates of (x_s, y_s) within the image reference frame, denoted (u, v), are such that:

$$\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f_u \frac{x_c}{z_c} + c_u} \\ \boldsymbol{f_v \frac{y_c}{z_c} + c_v} \end{pmatrix}$$

With:

- $f_u = \frac{i_w}{s_w} f$ and $f_v = \frac{i_h}{s_h} f$
- $c_u = i_w \left(c_x + \frac{1}{2}\right)$ and $c_v = i_h \left(c_y + \frac{1}{2}\right)$

Pinhole Camera

- **Definition:** Let a camera made of a sensor with a width s_w and a height s_h that produce an image with a width i_w and a height i_h. **The width p_w and the height p_h of an image pixel** are such that:

$$p_w = \frac{s_w}{i_w} \qquad p_h = \frac{s_h}{i_h}$$

Pinhole Camera

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$$p_w = \frac{s_w}{i_w} \qquad p_h = \frac{s_h}{i_h}$$

- If the image ratio and the sensor ratio are the same:

$$\frac{i_w}{i_h} = \frac{s_w}{s_h}$$

Pinhole Camera

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$$p_w = \frac{s_w}{i_w} \qquad p_h = \frac{s_h}{i_h}$$

- If the image ratio and the sensor ratio are the same:

$$\frac{i_w}{i_h} = \frac{s_w}{s_h} \longrightarrow s_w = \frac{i_w s_h}{i_h}$$

Pinhole Camera

- **Definition:** Let a camera made of a sensor with a width s_w and a height s_h that produce an image with a width i_w and a height i_h. **The width p_w and the height p_h of an image pixel** are such that:

$$p_w = \frac{s_w}{i_w} \qquad p_h = \frac{s_h}{i_h}$$

- If the image ratio and the sensor ratio are the same:

$$\frac{i_w}{i_h} = \frac{s_w}{s_h} \longrightarrow s_w = \frac{i_w s_h}{i_h} \longrightarrow p_w = \frac{s_w}{i_w} = \frac{i_w s_h}{i_h i_w} = \frac{s_h}{i_h} = p_h$$

Camera has square pixels

Pinhole Camera

- Let the computation of image projection of a 3D point:

$$\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f_u} \dfrac{\boldsymbol{x_c}}{\boldsymbol{z_c}} + \boldsymbol{c_u} \\ \boldsymbol{f_v} \dfrac{\boldsymbol{y_c}}{\boldsymbol{z_c}} + \boldsymbol{c_v} \end{pmatrix}$$

- **Notation:** f_u is called horizontal focal length with $f_u = \dfrac{i_w}{s_w} f = \dfrac{f}{p_w}$

- **Notation:** f_v is called vertical focal length with $f_v = \dfrac{i_h}{s_h} f = \dfrac{f}{p_h}$

Pinhole Camera

- Homogeneous computation of (u, v)

$$\alpha \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f_u & 0 & c_u & 0 \\ 0 & f_v & c_v & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix}$$

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- Homogeneous computation of (u, v)

$$\alpha \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f_u & 0 & c_u & 0 \\ 0 & f_v & c_v & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix}$$

Normalized perspective projection
(project onto an image plane located at $z = 1$)

Pinhole Camera

- Homogeneous computation of (u, v)

$$\alpha \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f_u & 0 & c_u & 0 \\ 0 & f_v & c_v & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix}$$

Normalized perspective projection
(project onto an image plane located at $z = 1$)

Camera reference frame
to image reference frame

Pinhole Camera

- Homogeneous computation of (u, v)

$$\alpha\begin{bmatrix}u\\v\\1\end{bmatrix}=\begin{bmatrix}f_u&0&c_u&0\\0&f_v&c_v&0\\0&0&1&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&0\end{bmatrix}\begin{bmatrix}x_c\\y_c\\z_c\\1\end{bmatrix}=\begin{bmatrix}f_u&0&c_u&0\\0&f_v&c_v&0\\0&0&1&0\end{bmatrix}\begin{bmatrix}x_c\\y_c\\z_c\\z_c\end{bmatrix}$$

Normalized perspective projection
onto image plane

Pinhole Camera

- Homogeneous computation of (u, v)

$$\alpha \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f_u & 0 & c_u & 0 \\ 0 & f_v & c_v & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \begin{bmatrix} f_u & 0 & c_u & 0 \\ 0 & f_v & c_v & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ z_c \end{bmatrix} = \begin{bmatrix} f_u x_c + c_u z_c \\ f_v y_c + c_v z_c \\ z_c \end{bmatrix}$$

Changing to image reference frame

Pinhole Camera

- Homogeneous computation of (u, v)

$$\alpha \begin{bmatrix}u\\v\\1\end{bmatrix} = \begin{bmatrix} f_u & 0 & c_u & 0\\ 0 & f_v & c_v & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&0\end{bmatrix}\begin{bmatrix}x_c\\y_c\\z_c\\1\end{bmatrix} = \begin{bmatrix} f_u & 0 & c_u & 0\\ 0 & f_v & c_v & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_c\\y_c\\z_c\\z_c\end{bmatrix} = \begin{bmatrix} f_u x_c + c_u z_c\\ f_v y_c + c_v z_c\\ z_c\end{bmatrix}$$

$$= \frac{1}{z_c}\begin{bmatrix}\frac{f_u x_c}{z_c} + c_u\\ \frac{f_v y_c}{z_c} + c_v\\ 1\end{bmatrix}$$

Factorization

Pinhole Camera

- Homogeneous computation of (u, v)

$$\alpha\begin{bmatrix}u\\v\\1\end{bmatrix}=\begin{bmatrix}f_u & 0 & c_u & 0\\0 & f_v & c_v & 0\\0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_c\\y_c\\z_c\\1\end{bmatrix}=\begin{bmatrix}f_u & 0 & c_u & 0\\0 & f_v & c_v & 0\\0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_c\\y_c\\z_c\\z_c\end{bmatrix}=\begin{bmatrix}f_u x_c + c_u z_c\\f_v y_c + c_v z_c\\z_c\end{bmatrix}$$

$$=\frac{1}{z_c}\begin{bmatrix}\frac{f_u x_c}{z_c}+c_u\\\frac{f_v y_c}{z_c}+c_v\\1\end{bmatrix}=\frac{1}{z_c}\begin{bmatrix}f_u\frac{x_c}{z_c}+c_u\\f_v\frac{y_c}{z_c}+c_v\\1\end{bmatrix}$$

Rewriting

Pinhole Camera

- Homogeneous computation of (u, v)

$$\alpha \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f_u & 0 & c_u & 0 \\ 0 & f_v & c_v & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \begin{bmatrix} f_u & 0 & c_u & 0 \\ 0 & f_v & c_v & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ z_c \end{bmatrix} = \begin{bmatrix} f_u x_c + c_u z_c \\ f_v y_c + c_v z_c \\ z_c \end{bmatrix}$$

$$= \frac{1}{z_c} \begin{bmatrix} \frac{f_u x_c}{z_c} + c_u \\ \frac{f_v y_c}{z_c} + c_v \\ 1 \end{bmatrix} = \frac{1}{z_c} \begin{bmatrix} f_u \frac{x_c}{z_c} + c_u \\ f_v \frac{y_c}{z_c} + c_v \\ 1 \end{bmatrix} = \begin{bmatrix} f_u \frac{x_c}{z_c} + c_u \\ f_v \frac{y_c}{z_c} + c_v \\ 1 \end{bmatrix}$$

$$\alpha X = X$$

Pinhole Camera

- Homogeneous computation of (u, v)

$$\alpha \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f_u & 0 & c_u & 0 \\ 0 & f_v & c_v & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \begin{bmatrix} f_u & 0 & c_u & 0 \\ 0 & f_v & c_v & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ z_c \end{bmatrix} = \begin{bmatrix} f_u x_c + c_u z_c \\ f_v y_c + c_v z_c \\ z_c \end{bmatrix}$$

$$= \frac{1}{z_c} \begin{bmatrix} \frac{f_u x_c}{z_c} + c_u \\ \frac{f_v y_c}{z_c} + c_v \\ 1 \end{bmatrix} = \frac{1}{z_c} \begin{bmatrix} f_u \frac{x_c}{z_c} + c_u \\ f_v \frac{y_c}{z_c} + c_v \\ 1 \end{bmatrix} = \begin{bmatrix} f_u \frac{x_c}{z_c} + c_u \\ f_v \frac{y_c}{z_c} + c_v \\ 1 \end{bmatrix} \equiv \begin{pmatrix} f_u \frac{x_c}{z_c} + c_u \\ f_v \frac{y_c}{z_c} + c_v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

Homogeneous to Euclidean

Pinhole Camera

- **Definition:** The pinhole relative projection that enable to project a 3D point (x_c, y_c, z_c) expressed within the camera reference frame to a 2D point (u, v) expressed within the image reference frame is the homogeneous transformation such as:

$$\alpha \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f_u & 0 & c_u & 0 \\ 0 & f_v & c_v & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix}$$

Terrain

- The 3D space to be projected through a camera is not necessarily linked to the camera's reference frame. Most of the time, the camera localization and the 3D space are positioned in a global reference frame.

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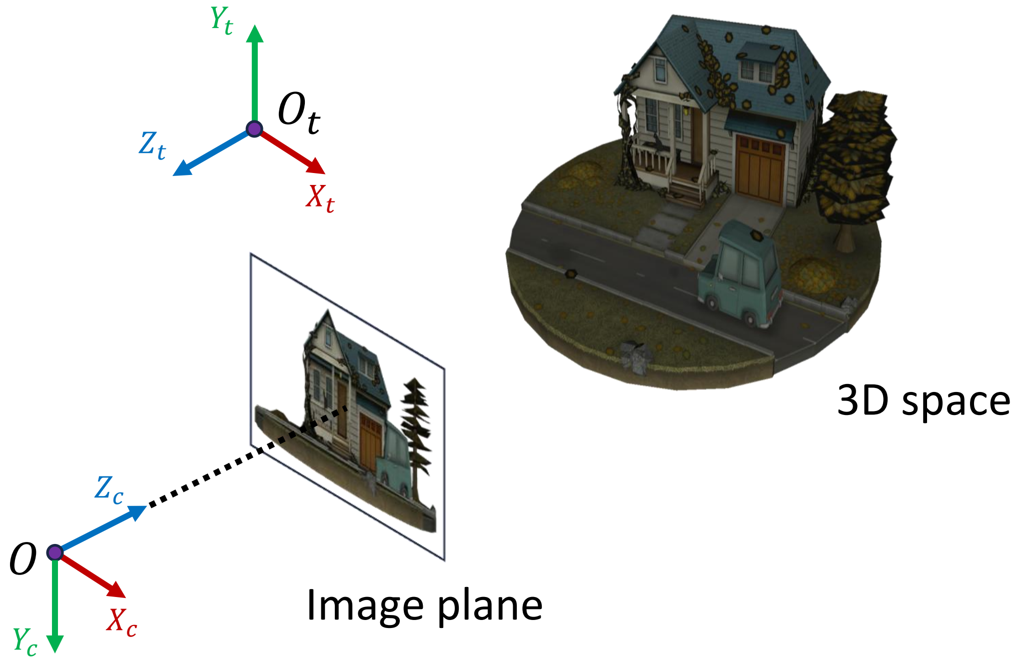
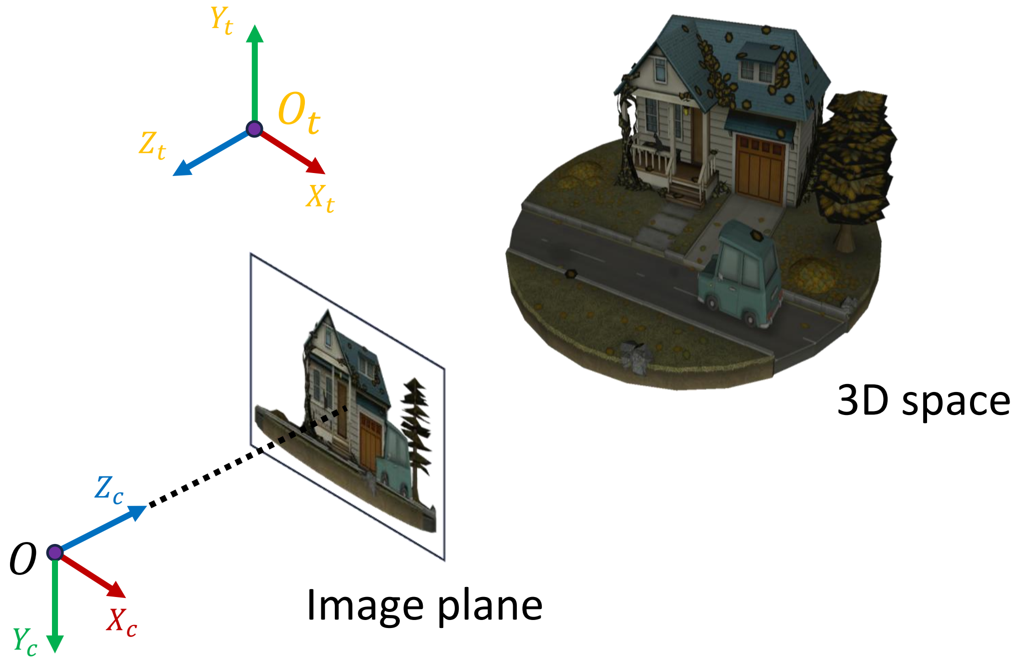


Terrain

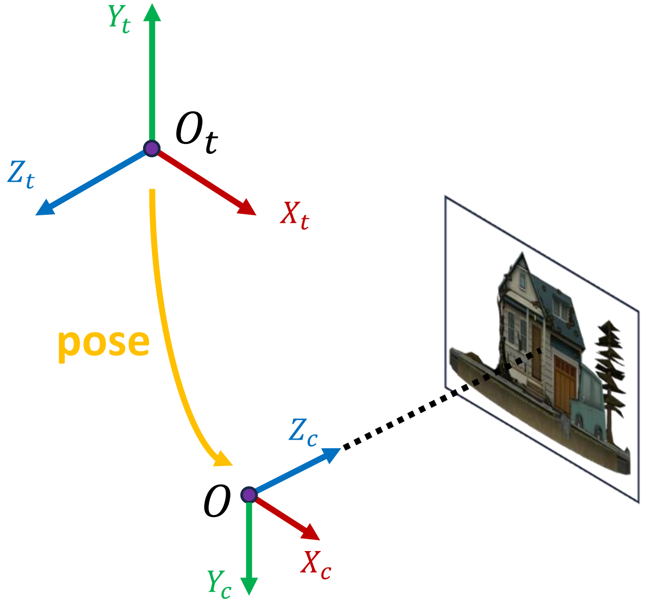
- **Definition:** The global reference frame within which the camera and 3D space are represented is called the **terrain reference frame**. And is denoted $\langle O_t, X_t, Y_t, Z_t \rangle$



Terrain

- Exists a composed homogeneous transformation that enables to transform the terrain reference frame into the camera reference frame:

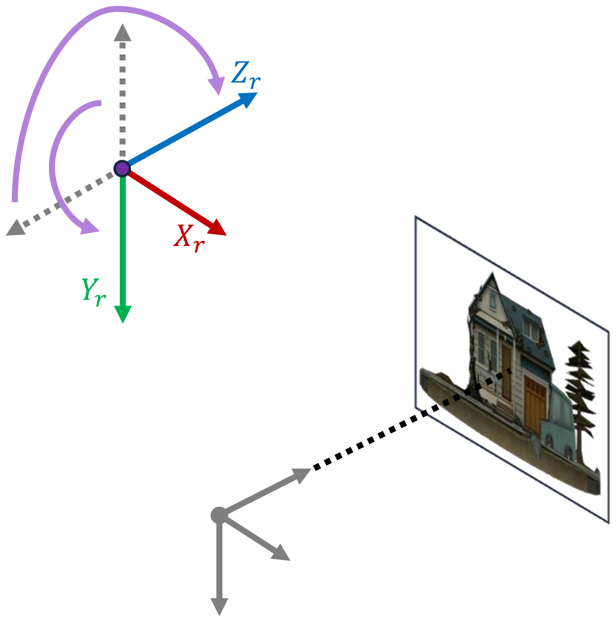
$$\begin{bmatrix}1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}$$



Terrain

- Exists a composed homogeneous transformation that enables to transform the terrain reference frame into the camera reference frame:

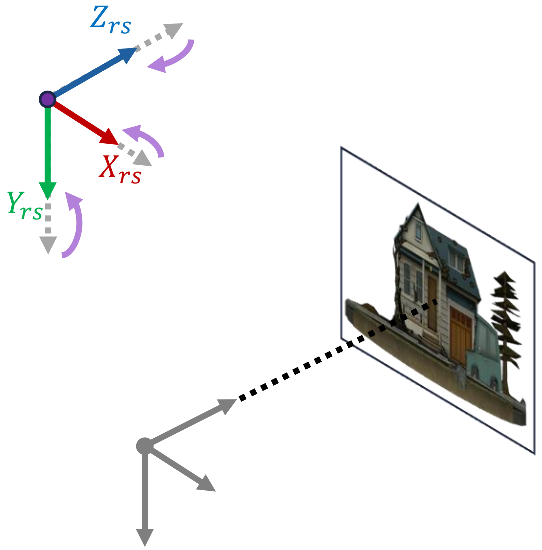
$$\begin{bmatrix}1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}$$



Terrain

- Exists a composed homogeneous transformation that enables to transform the terrain reference frame into the camera reference frame:

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



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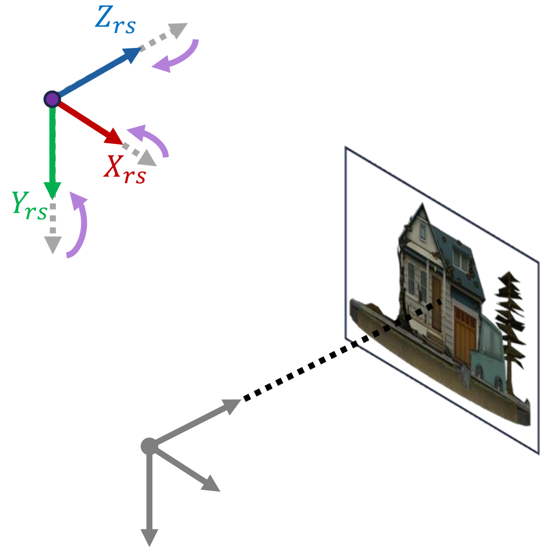
$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



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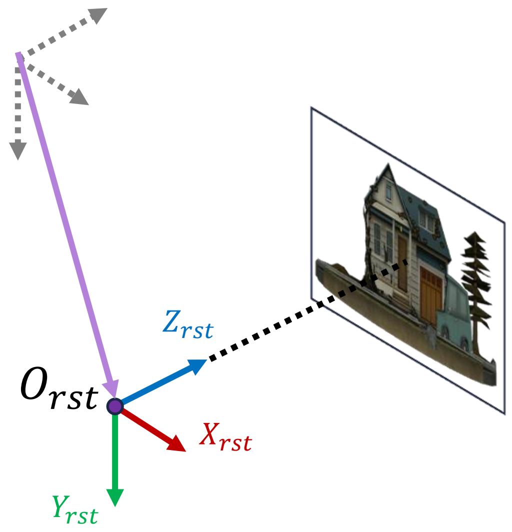
$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Terrain

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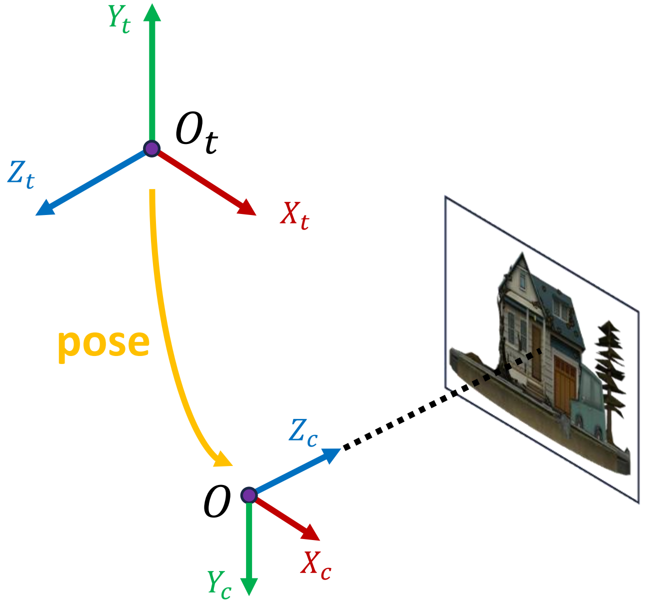
$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

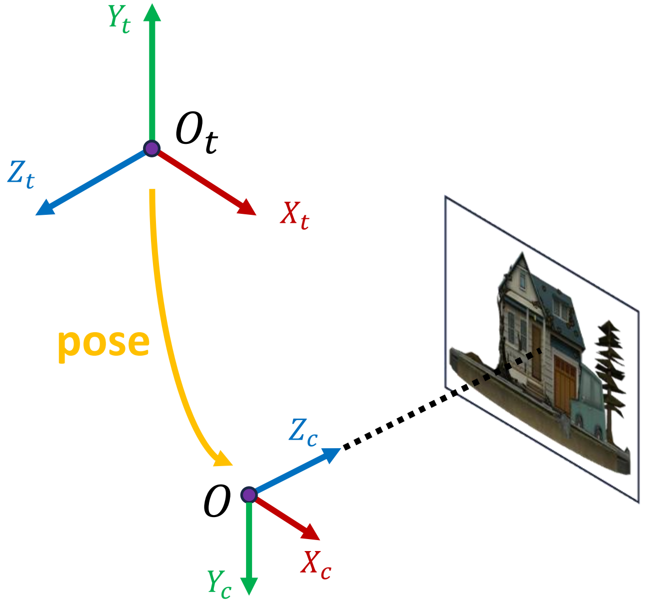
Matrix Product



Terrain

- Exists a composed homogeneous transformation that enables to transform the terrain reference frame into the camera reference frame:

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x r_{11} & s_x r_{12} & s_x r_{13} & 0 \\ s_y r_{21} & s_y r_{22} & s_y r_{23} & 0 \\ s_z r_{31} & s_z r_{32} & s_z r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



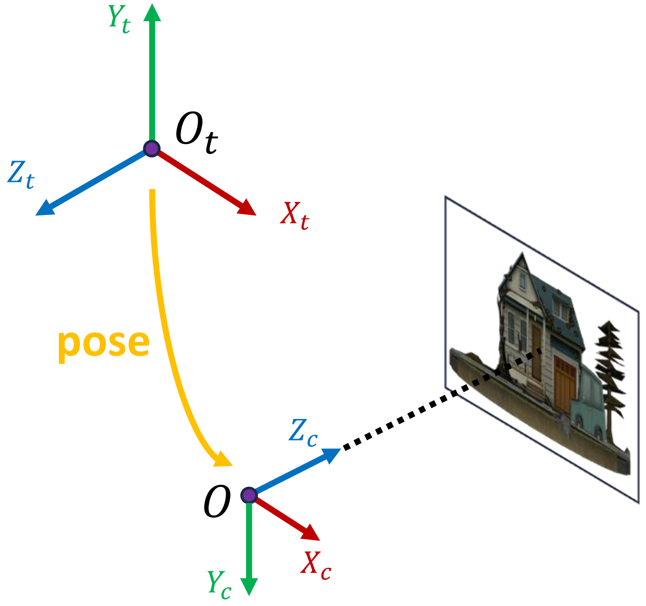
Photogrammetry / Definition & Principles

Terrain

- Exists a composed homogeneous transformation that enables to transform the terrain reference frame into the camera reference frame:

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x r_{11} & s_x r_{12} & s_x r_{13} & 0 \\ s_y r_{21} & s_y r_{22} & s_y r_{23} & 0 \\ s_z r_{31} & s_z r_{32} & s_z r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

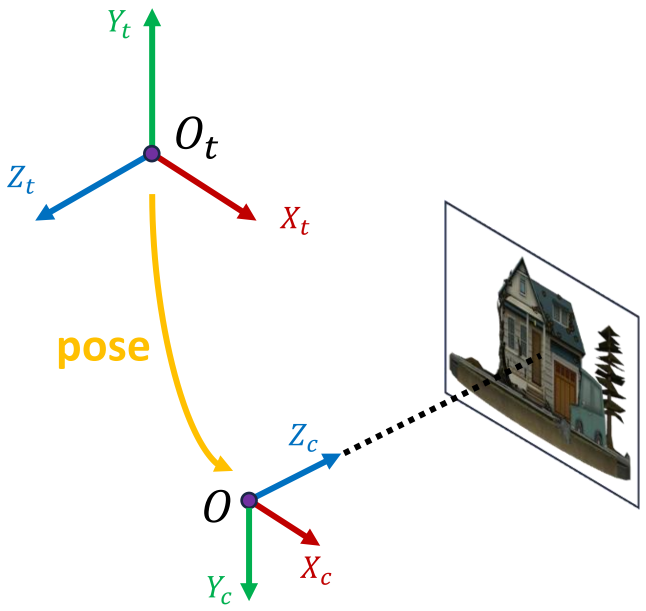
Matrix Product



Terrain

- Exists a composed homogeneous transformation that enables to transform the terrain reference frame into the camera reference frame:

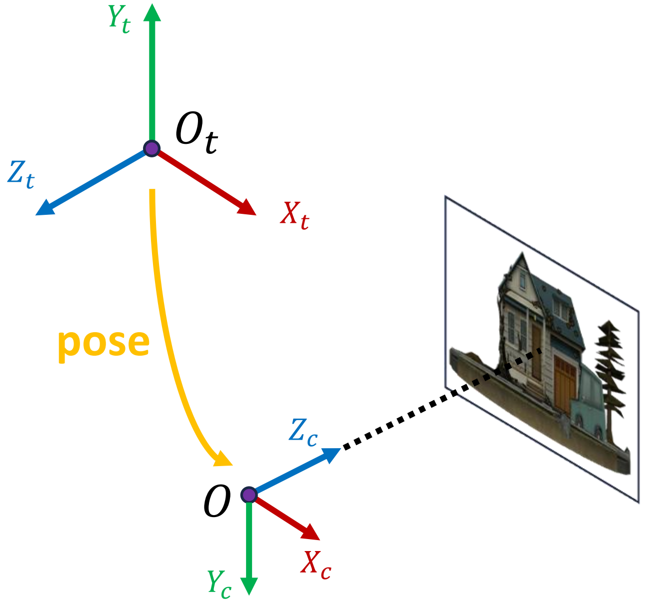
$$\begin{bmatrix} s_x r_{11} & s_x r_{12} & s_x r_{13} & t_x \\ s_y r_{21} & s_y r_{22} & s_y r_{23} & t_y \\ s_z r_{31} & s_z r_{32} & s_z r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Terrain

- **Definition:** The transformation that enables to transform the terrain reference frame into the camera reference frame is called **pose** and is represented by the homogeneous transformation:

$$\begin{bmatrix} s_x r_{11} & s_x r_{12} & s_x r_{13} & t_x \\ s_y r_{21} & s_y r_{22} & s_y r_{23} & t_y \\ s_z r_{31} & s_z r_{32} & s_z r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Pinhole Camera

- **Definition:** The pinhole global projection that enable to project a 3D point (x_t, y_t, z_t) expressed within the terrain reference frame to a 2D point (u, v) expressed within the image reference frame is the homogeneous transformation such as:

$$\alpha \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f_u & 0 & c_u & 0 \\ 0 & f_v & c_v & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s_x r_{11} & s_x r_{12} & s_x r_{13} & t_x \\ s_y r_{21} & s_y r_{22} & s_y r_{23} & t_y \\ s_z r_{31} & s_z r_{32} & s_z r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \\ 1 \end{bmatrix}$$

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$$\alpha \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f_u & 0 & c_u & 0 \\ 0 & f_v & c_v & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s_x r_{11} x_t + s_x r_{12} y_t + s_x r_{13} z_t + t_x \\ s_y r_{21} x_t + s_y r_{22} y_t + s_y r_{23} z_t + t_y \\ s_z r_{31} x_t + s_z r_{32} y_t + s_z r_{33} z_t + t_z \\ 1 \end{bmatrix}$$

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$$\alpha \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f_u(s_x r_{11} x_t + s_x r_{12} y_t + s_x r_{13} z_t + t_x) + c_u(s_z r_{31} x_t + s_z r_{32} y_t + s_z r_{33} z_t + t_z) \\ f_v\left(s_y r_{21} x_t + s_y r_{22} y_t + s_y r_{23} z_t + t_y\right) + c_v(s_z r_{31} x_t + s_z r_{32} y_t + s_z r_{33} z_t + t_z) \\ s_z r_{31} x_t + s_z r_{32} y_t + s_z r_{33} z_t + t_z \end{bmatrix}$$

Pinhole Camera

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$$\alpha \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \frac{1}{s_z r_{31} x_t + s_z r_{32} y_t + s_z r_{33} z_t + t_z} \begin{bmatrix} \frac{f_u\left(s_x r_{11} x_t + s_x r_{12} y_t + s_x r_{13} z_t + t_x\right)}{s_z r_{31} x_t + s_z r_{32} y_t + s_z r_{33} z_t + t_z} + c_u \\ \frac{f_v\left(s_y r_{21} x_t + s_y r_{22} y_t + s_y r_{23} z_t + t_y\right)}{s_z r_{31} x_t + s_z r_{32} y_t + s_z r_{33} z_t + t_z} + c_v \\ 1 \end{bmatrix}$$

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$$\alpha \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \dfrac{f_u(s_x r_{11} x_t + s_x r_{12} y_t + s_x r_{13} z_t + t_x)}{s_z r_{31} x_t + s_z r_{32} y_t + s_z r_{33} z_t + t_z} + c_u \\ \dfrac{f_v(s_y r_{21} x_t + s_y r_{22} y_t + s_y r_{23} z_t + t_y)}{s_z r_{31} x_t + s_z r_{32} y_t + s_z r_{33} z_t + t_z} + c_v \\ 1 \end{bmatrix} \equiv \begin{pmatrix} \dfrac{f_u(s_x r_{11} x_t + s_x r_{12} y_t + s_x r_{13} z_t + t_x)}{s_z r_{31} x_t + s_z r_{32} y_t + s_z r_{33} z_t + t_z} + c_u \\ \dfrac{f_v(s_y r_{21} x_t + s_y r_{22} y_t + s_y r_{23} z_t + t_y)}{s_z r_{31} x_t + s_z r_{32} y_t + s_z r_{33} z_t + t_z} + c_v \end{pmatrix}$$

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Camera

- Collinearity Equations (from matrix product)

$$u = f_x \frac{r_{11}x + r_{12}y + r_{13}z + t_x}{r_{31}x + r_{32}y + r_{33}z + t_z} + c_x$$

$$v = f_y \frac{r_{21}x + r_{22}y + r_{23}z + t_y}{r_{31}x + r_{32}y + r_{33}z + t_z} + c_y$$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- 18 variables equations system
  - 16 variables if projections are known (u, v)

Camera

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$$u = f_x \frac{r_{11}x + r_{12}y + r_{13}z + t_x}{r_{31}x + r_{32}y + r_{33}z + t_z} + c_x$$

$$v = f_y \frac{r_{21}x + r_{22}y + r_{23}z + t_y}{r_{31}x + r_{32}y + r_{33}z + t_z} + c_y$$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- 18 variables equations system
  - 16 variables if projections are known (u, v)
  - 13 variables if 3D positions are also known (x, y, z)

Calibration

- Fixing variables within the equations

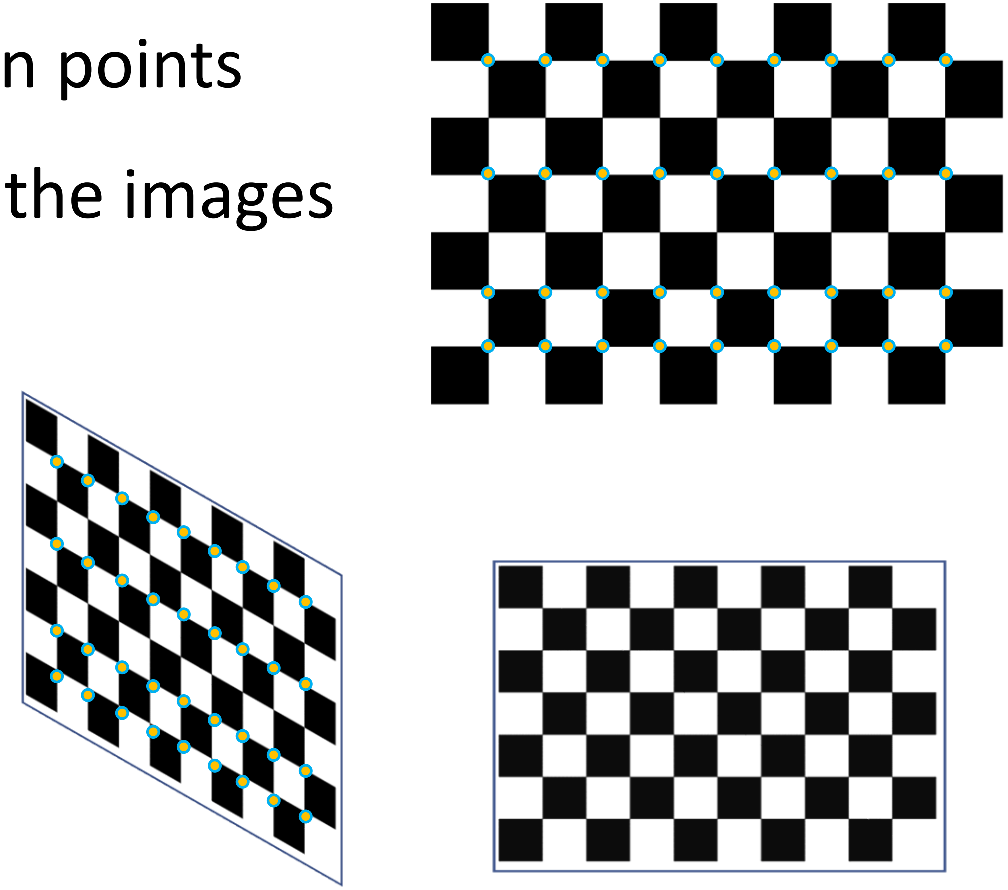
$$u = f_x \frac{r_{11}x + r_{12}y + r_{13}z + t_x}{r_{31}x + r_{32}y + r_{33}z + t_z} + c_x$$

$$v = f_y \frac{r_{21}x + r_{22}y + r_{23}z + t_y}{r_{31}x + r_{32}y + r_{33}z + t_z} + c_y$$

- How to ?
  - Taking images of an object with known points
  - Determining the point position within the images

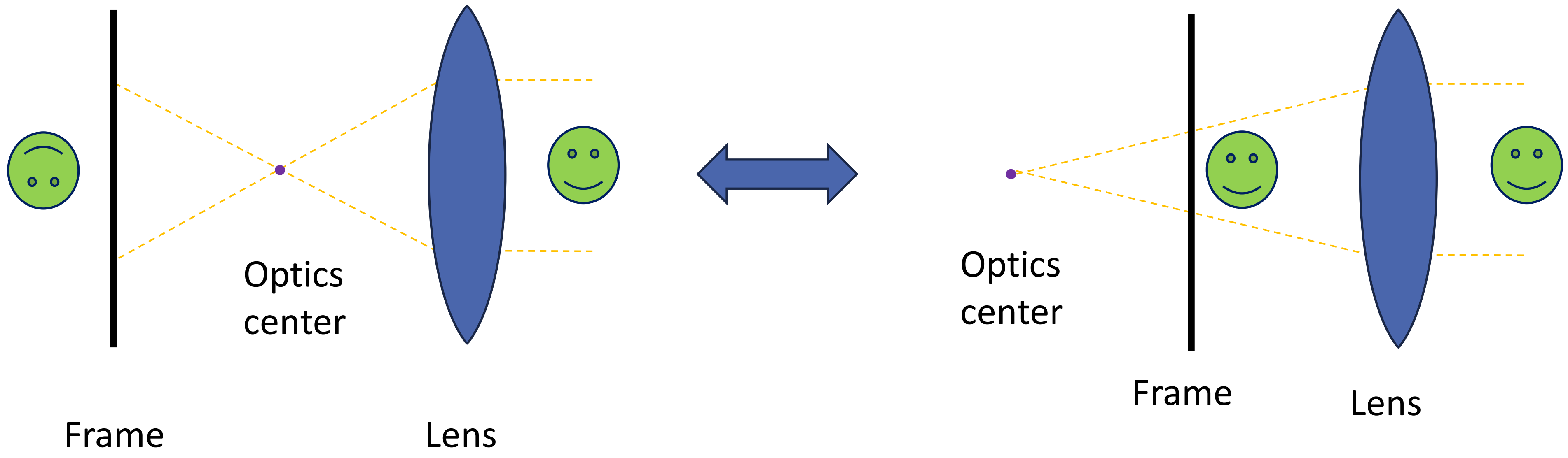
Calibration

- How to ?
  - Taking images of an object with known points
  - Determining the point position within the images



Physical camera

- Camera can be represented as a lens and a frame

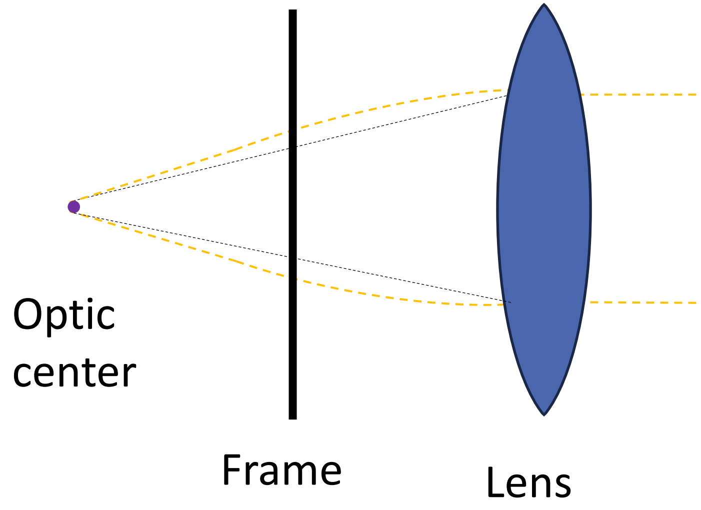


Physical representation

Virtual representation

Camera distortion

- Radial distortion
  - Light rays bend near the edges of a lens

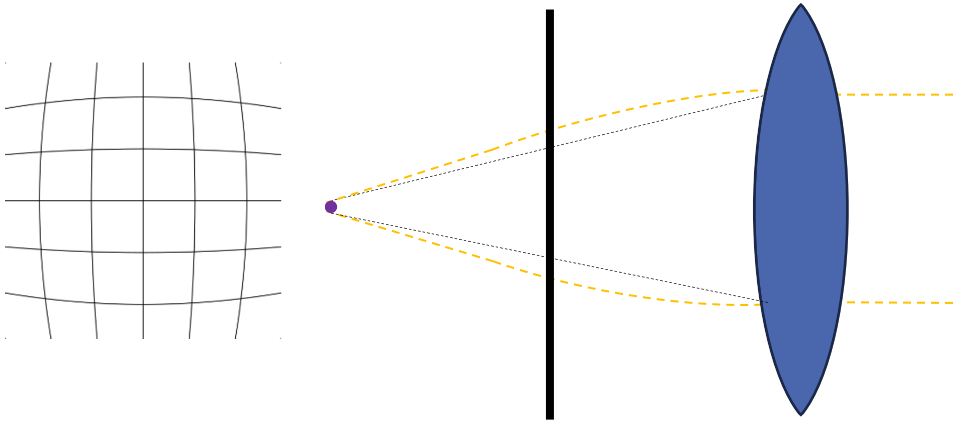


Radial distorted image

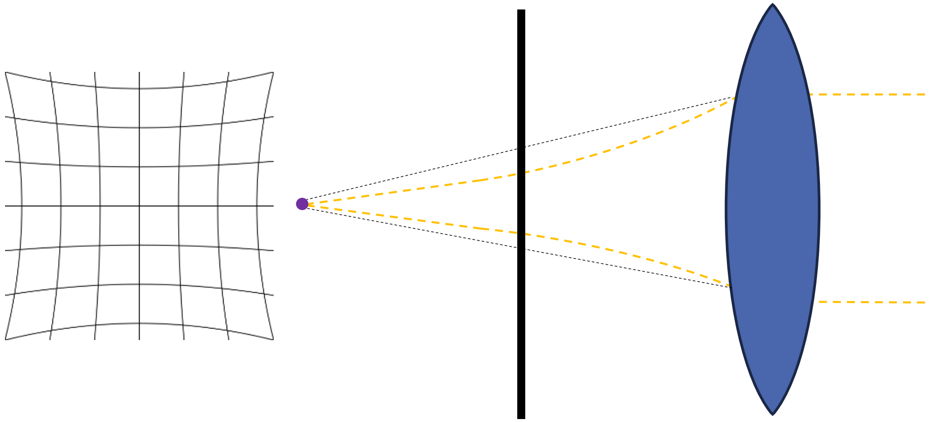
Original view

Camera distortion

- Radial distortion
  - Two main types



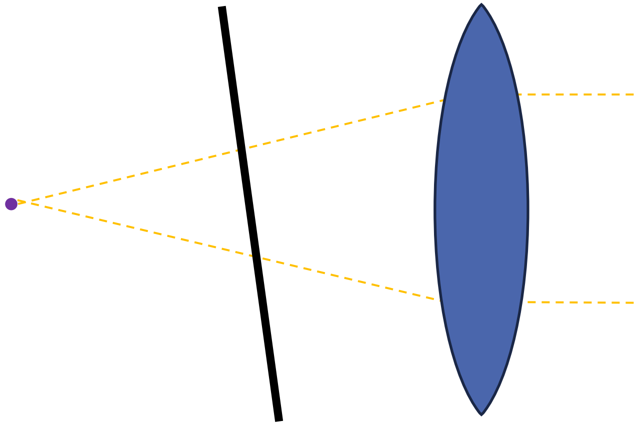
Barrel Distortion



Pincushion Distortion

Camera distortion

- Tangential distortion
  - Occurs when the lens and the image plane are not parallel



Tangential distorted image

Original view

Camera distortion

- Brown’s distortion model
  - Represents radial and tangential distortion with polynomials

$$x_d = u(1 + K_1 r^2 + K_2 r^4 + \cdots + K_n r^{2n}) + (P_2(r^2 + 2u^2) + 2P_1 uv)\left(1 + P_3 r^2 + \cdots + P_m r^{2(m-2)}\right)$$

$$y_d = v(1 + K_1 r^2 + K_2 r^4 + \cdots + K_n r^{2n}) + (P_1(r^2 + 2v^2) + 2P_2 uv)\left(1 + P_3 r^2 + \cdots + P_m r^{2(m-2)}\right)$$

Where:

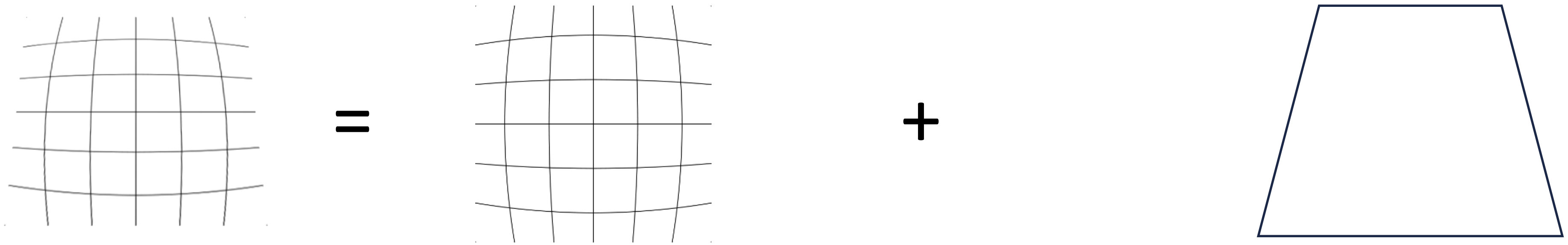
- u, v are the coordinates of a point on the frame without any distortion
- K_i are the radial distortion parameters
- P_j are the tangential distortion parameters
- r is the distance of the (u, v) point from the frame center

Camera distortion

- Brown's distortion model
  - Represents radial and tangential distortion with polynomials

$$x_d = u(1 + K_1 r^2 + K_2 r^4 + \cdots + K_n r^{2n}) + (P_2(r^2 + 2u^2) + 2P_1 uv)\left(1 + P_3 r^2 + \cdots + P_m r^{2(m-2)}\right)$$

$$y_d = \underbrace{v(1 + K_1 r^2 + K_2 r^4 + \cdots + K_n r^{2n})}_{\text{Radial distortion}} + \underbrace{(P_1(r^2 + 2v^2) + 2P_2 uv)\left(1 + P_3 r^2 + \cdots + P_m r^{2(m-2)}\right)}_{\text{Tangential distortion}}$$



Camera distortion

- Brown’s distortion model
  - Most of the time, K and P parameters number are limited

$$x_d = u(1 + K_1 r^2 + K_2 r^4 + K_3 r^6) + (P_2(r^2 + 2u^2) + 2P_1 uv)$$

$$y_d = v(1 + K_1 r^2 + K_2 r^4 + K_3 r^6) + (P_1(r^2 + 2v^2) + 2P_2 uv)$$

Camera distortion

- Complete representation

$$u = f_x \frac{r_{11}x + r_{12}y + r_{13}z + t_x}{r_{31}x + r_{32}y + r_{33}z + t_z} + c_x$$

$$v = f_y \frac{r_{21}x + r_{22}y + r_{23}z + t_y}{r_{31}x + r_{32}y + r_{33}z + t_z} + c_y$$

$$Camera\ Matrix = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$Distortion = [K_1 \quad K_2 \quad K_3 \quad P_1 \quad P_2]$$

$$x_d = u(1 + K_1r^2 + K_2r^4 + K_3r^6) + (P_2(r^2 + 2u^2) + 2P_1uv)$$

$$y_d = v(1 + K_1r^2 + K_2r^4 + K_3r^6) + (P_1(r^2 + 2v^2) + 2P_2uv)$$

Camera distortion

- Exercise

From the website:

http://web.seinturier.fr/teaching/computer-vision

Perform the Practical Work:

PW 4P: Camera calibration with OpenCV/Python

Image projection

- Let a camera defined by:
  - A referential $\mathcal{R}_c = \langle O_c, X_c, Y_c, Z_c \rangle$
  - A projection plane $\mathcal{P}$ located at a distance f of oc along zc
- The projection matrix P of the camera is such that:

$$\mathrm{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix}$$

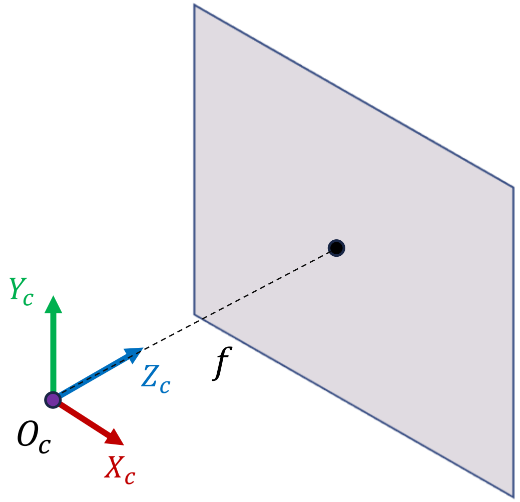


Image projection

- Let a camera defined by:
  - A referential $\mathcal{R}_C = \langle O_c, X_c, Y_c, Z_c \rangle$
  - A projection plane $\mathcal{P}$ located at a distance f of oc along zc
- Let $\psi = (x, y, z)$ a point within $\mathcal{R}_c$ and psi its homogeneous representation

$$\psi_p = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} fx/z \\ fy/z \\ f \end{pmatrix}$$

$$\Psi = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\mathrm{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix}$$

$$\Psi_p = \begin{bmatrix} x_p \\ y_p \\ z_p \\ 1 \end{bmatrix} = \mathrm{P}\Psi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ z/f \end{bmatrix} = \begin{bmatrix} fx/z \\ fy/z \\ f \\ 1 \end{bmatrix}$$

$$\psi = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

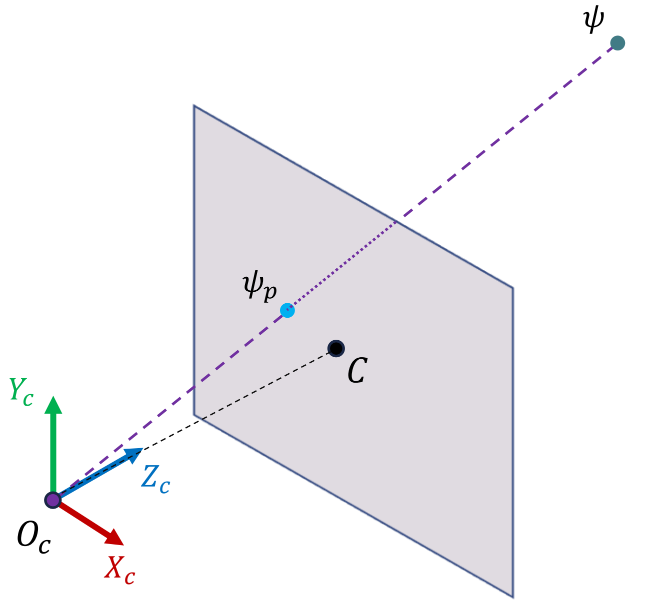
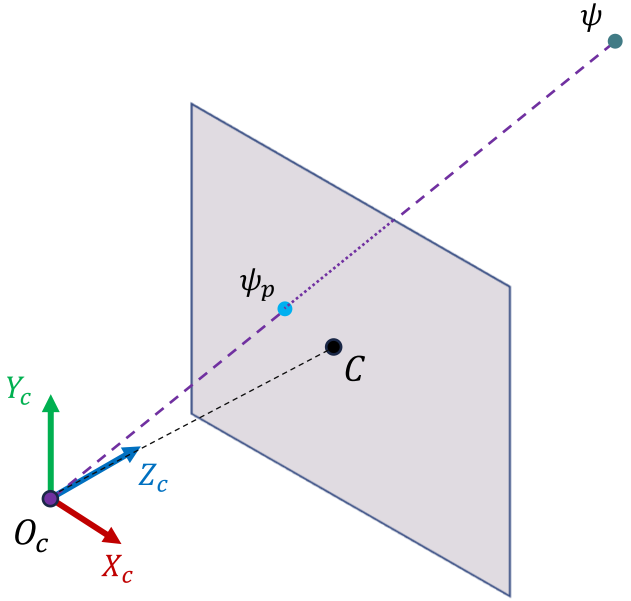


Image projection

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \qquad \psi = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \Psi = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\Psi_p = \begin{bmatrix} x_p \\ y_p \\ z_p \\ 1 \end{bmatrix} = P\Psi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ z/f \end{bmatrix} = \begin{bmatrix} fx/z \\ fy/z \\ f \\ 1 \end{bmatrix}$$

$$\psi_p = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} fx/z \\ fy/z \\ f \end{pmatrix}$$

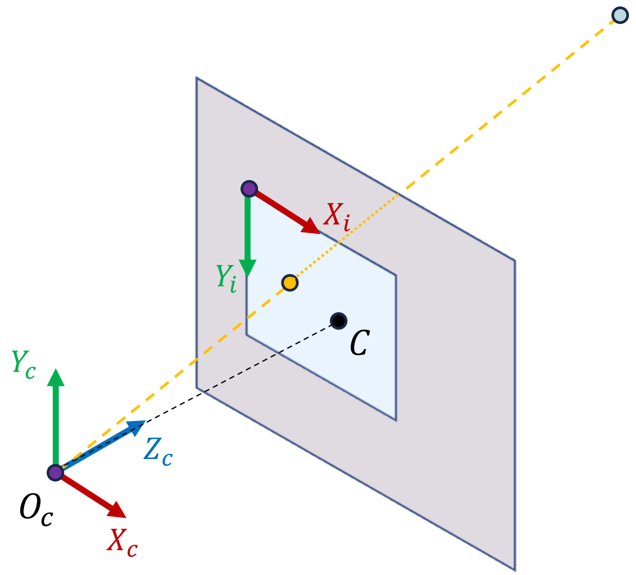


Frame projection

$$\mathrm{K} = \begin{bmatrix}1 & 0 & 0 & c_x\\0 & 1 & 0 & c_y\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}k_u & 0 & 0 & 0\\0 & k_v & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\0 & -1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}k_u & 0 & 0 & c_x\\0 & -k_v & 0 & c_y\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

$$\mathrm{K}\Psi' = \begin{bmatrix}k_u & 0 & 0 & c_x\\0 & -k_v & 0 & c_y\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}fx/z\\fy/z\\f\\1\end{bmatrix} =$$

$$\mathrm{K}\Psi' = \begin{bmatrix}k_u & 0 & 0 & c_x\\0 & -k_v & 0 & c_y\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}fx/z\\fy/z\\f\\1\end{bmatrix} = \begin{bmatrix}k_u fx/z + c_x\\-k_v fy/z + c_y\\1\end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{K} = \begin{bmatrix} k_u & 0 & 0 & 0 \\ 0 & k_v & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} k_u & 0 & 0 & 0 \\ 0 & -k_v & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{K}\Psi_p = \begin{bmatrix} k_u & 0 & 0 & c_x \\ 0 & -k_v & 0 & c_y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} fx/z \\ fy/z \\ f \\ 1 \end{bmatrix} = \begin{bmatrix} k_u fx/z + c_x \\ -k_v fx/z + c_y \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} k_u & 0 & 0 & 0 \\ 0 & -k_v & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{K} = \begin{bmatrix} 1 & 0 & 0 & c_x \\ 0 & 1 & 0 & c_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} k_u & 0 & 0 & c_x \\ 0 & -k_v & 0 & c_y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\alpha\begin{bmatrix}x_c\\y_c\\z_c\\1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1/f&0\end{bmatrix}\begin{bmatrix}x_c\\y_c\\z_c\\1\end{bmatrix}=\begin{bmatrix}x_c\\y_c\\z_c\\z_c/f\end{bmatrix}=\frac{z_c}{f}\begin{bmatrix}fx_c/z_c\\fy_c/z_c\\f\\1\end{bmatrix}=\begin{bmatrix}fx_c/z_c\\fy_c/z_c\\f\\1\end{bmatrix}\rightarrow\begin{pmatrix}fx_c/z_c\\fy_c/z_c\\f\end{pmatrix}=\begin{pmatrix}x_p\\y_p\\z_p\end{pmatrix}$$

$$\alpha\begin{bmatrix}x_c\\y_c\\z_c\\1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1/f&0\end{bmatrix}\begin{bmatrix}x_c\\y_c\\z_c\\1\end{bmatrix}=\begin{bmatrix}x_c\\y_c\\z_c\\z_c/f\end{bmatrix}=\frac{z_c}{f}\begin{bmatrix}fx_c/z_c\\fy_c/z_c\\f\\1\end{bmatrix}$$

$$\begin{bmatrix}\frac{i_w}{s_w}&0&0&0\\0&\frac{i_h}{s_h}&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}fx_c/z_c\\fy_c/z_c\\f\\1\end{bmatrix}=\begin{bmatrix}\frac{i_w}{s_w}\frac{fx_c}{z_c}\\\frac{i_h}{s_h}\frac{fx_c}{z_c}\\0\\1\end{bmatrix}=\begin{bmatrix}f\frac{i_w}{s_w}\frac{fx_c}{z_c}\\f\frac{i_h}{s_h}\frac{fx_c}{z_c}\\0\\1\end{bmatrix}=\begin{bmatrix}f_x\frac{x_c}{z_c}\\f_y\frac{x_c}{z_c}\\0\\1\end{bmatrix}$$

$$\alpha \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \begin{bmatrix} x_c \\ y_c \\ z_c \\ z_c/f \end{bmatrix} = \frac{z_c}{f} \begin{bmatrix} f x_c/z_c \\ f y_c/z_c \\ f \\ 1 \end{bmatrix} = \begin{bmatrix} f x_c/z_c \\ f y_c/z_c \\ f \\ 1 \end{bmatrix} \rightarrow \begin{pmatrix} f x_c/z_c \\ f y_c/z_c \\ f \end{pmatrix} = \begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix}$$

$$\alpha \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \begin{bmatrix} x_c \\ y_c \\ z_c \\ z_c/f \end{bmatrix} = \frac{z_c}{f} \begin{bmatrix} f x_c/z_c \\ f y_c/z_c \\ f \\ 1 \end{bmatrix}$$

$$\begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix} = \begin{pmatrix} \dfrac{f x_c}{z_c} \\ \dfrac{f x_c}{z_c} \\ f \end{pmatrix}$$

$$\begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix} = \begin{pmatrix} f x_c/z_c \\ f y_c/z_c \\ f \end{pmatrix}$$

$$\begin{bmatrix} s_x r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & s_y r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & s_z r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} s_x r_{11} & s_x r_{12} & s_x r_{13} & 0 \\ s_y r_{21} & s_y r_{22} & s_y r_{23} & 0 \\ s_z r_{31} & s_z r_{32} & s_z r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} s_x r_{11} & s_x r_{12} & s_x r_{13} & t_x \\ s_y r_{21} & s_y r_{22} & s_y r_{23} & t_y \\ s_z r_{31} & s_z r_{32} & s_z r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$